1 Turán’s theorem

1.1 Statement & proof

I think the following proof is due to Alon and Spencer.

Theorem 1.1 (Turán’s theorem) Let \( G = (V, E) \) be a graph. The graph \( G \) has an independent set of size \( \frac{n}{1 + d_G} \), where \( n = |V| \) and \( d_G \) is the average vertex degree in \( G \).

Proof: Let \( \pi = (\pi_1, \ldots, \pi_n) \) be a random permutation of the vertices of \( G \). Pick the vertex \( \pi_i \) into the independent set if none of its neighbors appear before it in \( \pi \). Clearly, \( v \) appears in the independent set if and only if it appears in the permutation before all its \( d(v) \) neighbors. The probability for this is \( \frac{1}{1 + d(v)} \). Thus, the expected size of the independent set is (exactly)

\[
\tau = \sum_{v \in V} \frac{1}{1 + d(v)},
\]

by linearity of expectations. Thus, by the probabilistic method, there exists an independent set in \( G \) of size at least \( \tau \).

We remain with the task of proving that \( \tau \geq \frac{n}{1 + d_G} \). Observe that if \( x + y = \alpha \), then

\[
\frac{1}{1 + x} + \frac{1}{1 + y} = \frac{1 + x + 1 + y}{1 + x + y + xy} = \frac{2 + \alpha}{1 + \alpha + xy} \geq \frac{2 + \alpha}{1 + \alpha + \alpha^2/4} = \frac{2(1 + \alpha/2)}{(1 + \alpha/2)^2} = \frac{2}{1 + \alpha/2},
\]

since the quantity \( xy \) is maximized when \( x = y \) under the condition \( x + y = \alpha \). This implies that the minimum of Eq. (1) is achieved if we replace \( d(v) \) by the average degree in \( G \), which implies the theorem.

Following a post of this write-up on my blog, readers suggested two modifications. We present an alternative proof incorporating both suggestion.

Alternative proof of Theorem 1.1 We associate a charge of size \( 1/(d(v) + 1) \) with each vertex of \( G \). Let \( \gamma(G) \) denote the total charge of the vertices of \( G \). We prove, using induction, that there is always an independent set in \( G \) of size at least \( \gamma(G) \). If \( G \) is the empty graph, then the claim trivially holds. Otherwise, assume that it holds if the graph has at most \( n - 1 \) vertices, and consider the vertex \( v \) of lowest degree in \( G \). The total charge of \( v \) and its neighbors is

\[
\frac{1}{d(v) + 1} + \sum_{uv \in E} \frac{1}{d(u) + 1} \leq \frac{1}{d(v) + 1} + \sum_{uv \in E} \frac{1}{d(v) + 1} = \frac{d(v) + 1}{d(v) + 1} = 1,
\]

since \( d(u) \geq d(v) \), for all \( uv \in E \). Now, consider the graph \( H \) resulting from removing \( v \) and its neighbors from \( G \). Clearly, \( \gamma(H) \) is larger (or equal) to the total charge of the vertices of \( V(H) \) in \( G \), as their degree had either decreased (or remained the same). As such, by induction, we have an independent set in \( H \) of size at least \( \gamma(H) \). Together with \( v \) this forms an independent set in \( G \) of size at least \( \gamma(H) + 1 \geq \gamma(G) \). Implying that there exists an independent set in \( G \) of size

\[
\tau = \sum_{v \in V} \frac{1}{1 + d(v)},
\]

(2)
Now, set $x_v = 1 + d(v)$, and observe that

$$
(n + 2|E|) \tau = \left(\sum_{v \in V} x_v\right) \left(\sum_{v \in V} \frac{1}{x_v}\right) \geq \sum_{v \in V} x_v \frac{1}{x_v} = n.
$$

Namely, $\tau \geq \frac{n}{n + 2|E|} = \frac{1}{1 + 2|E|/n} = \frac{1}{1 + d_G}$.

### 1.2 An algorithm for the weighted case

In the weighted case, we associate weight $w(v)$ with each vertex of $G$, and we are interested in the maximum weight independent set in $G$. Deploying the algorithm described in the first proof of Theorem 1.1, implies the following.

**Lemma 1.2** The graph $G = (V, E)$ has an independent set of size $\geq \sum_{v \in V} \frac{w(v)}{1 + d(v)}$.

**Proof:** By linearity of expectations, we have that the expected weight of the independent set computed is equal to

$$
\sum_{v \in V} w(v) \cdot \Pr[v \text{ in the independent set}] = \sum_{v \in V} \frac{w(v)}{1 + d(v)}.
$$