Final Exam

December 9, 2010
CS 598shp - Randomized Algorithms - Fall 2010

Version: 1.03

Guidelines

Due date. You have to submit the exam by 12/15/10 23:59:59.

Submission. You need to submit the solution to this exam both electronically and as a printout. The printout should be put under the door of my office. Note, that you need to sign the printout (see below).

The electronic submission should done by email (to me) and include a latex file with all the problems solved (additional figures, if needed, should also included) and a pdf file. The latex file must pass latex without any error.

Honesty, etc

(I am sorry I have to put this in, but there were some outrageous incidences lately in other graduate classes, and I want to prevent such cases.)

This is a final exam, and should be solved accordingly. Namely, you have to do this exam on your own without any interaction with other students in the class (or outside the class). In particular, any source you use outside the class notes must be explicitly and carefully stated, including sources found on the web, people you discussed the problems with (but you should not be doing it, as noted above), etc. Failing to disclose such sources would be considered to be cheating, and would be handled according to the university/department policy.

In particular, by signing the line below you indicate that you understand and agree to these rules, and understand fully the implications, and that you had done this exam according to the above guidelines.

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Signature

1 The desperate balls and bins game.

You throw n balls into n bins. When throwing each ball, the following "game" is being played (as such, this game is going to be played n times).

In the ith round of the game (starting with $i = 1$), the algorithm picks $f(i)$ bins uniformly out of the n bins, say $b_1, \ldots, b_{f(i)}$, where $f(i)$ is some function. It then scans these bins one by one. The ball is thrown into the first bin in this list that has less than i balls in it. Otherwise, if the load (i.e., number of balls in a bin) of all these bins is i or more, then the algorithm continues to the next round of the game. Here is pseudo code of this:

```
algThrowBalls(n)
   bins_sizes[1...n] ← 0
   for k = 1, ..., n do
        algThrowASingleBall( k )


algThrowASingleBall( k )
   for i = 1, ..., ∞ do
        for j = 1, ..., f(i) do
             x ← random number in {1, ..., n}
             if bins_sizes[x] < i then
                  Put the ball k in the xth bin
                  bins_sizes[x] ← bins_sizes[x] + 1
                  return
```

(A) Prove (as tight as possible) upper and lower bounds on the maximum load of a bin (out of the n bins) under this scheme after n balls had been thrown in and $f(i) = i$. (The bound should hold with high probability.)

(B) Repeat the above, for the case where, in the ith iteration, the algorithm inspects $f(i) = k^i$ different bins. Here $k > 1$ is a small positive integer constant.

2 Shannon's Theorem.

Read and understand Shannon's theorem, described here:

`http://valis.cs.uiuc.edu/~sariel/teach/10/a_rand_alg/lec/28_entropy_III.pdf`

And answer the following question.

In proving Shannon's theorem, we used the following decoding method: Look for a codeword that differs from the received sequence of bits in between $(p - \varepsilon)n$ and $(p + \varepsilon)n$ places, for an appropriate choice of ε. If there is only one such codeword, the decoder concludes that this codeword was the one sent. Suppose instead that the decoder looks for the codeword that differs from the received sequence in the smallest number of bits (breaking ties arbitrarily), and concludes that this codeword was the one sent. Show (in detail!) how to modify the proof Shannon's theorem for this decoding technique to obtain a similar result.