# Homework 3, Due 23:59:59, 11/2/10 - slide it under the door of my office (SC 3306)

CS 598shp - Randomized Algorithms - Fall 2010

October 19, 2010

Version: 1.00

## Problems

1. **The exploding occupancy problem revisited. [60 Points]**

   You are throwing balls into $n$ bins. If a bin is not empty, you empty the bin (since the bin exploded from having too many balls in it). Namely, as soon as a bin has two or more balls, you throw away these balls and empty the bin (i.e., at any points in time, a bin is either empty or contains a single ball).

   (i) [**5 Points**] Prove that if you thrown in $n/4$ balls, then in expectation, at least $n/8$ bins are non empty in the end of throwing the balls.

   (ii) [**5 Points**] Prove that after $O(n^2)$ balls thrown into the bins, at some point there were (at least) $n/2$ non-empty bins, and this holds with (say) constant probability.

   (iii) [**50 Points**] Prove that after $O(n \log n)$ balls thrown into the bins, at some point there were (at least) $n/2$ non-empty bins, and this holds with high probability.

2. Divide and conquer.
   [**10 Points**]

   A standard technique for divide and conquer using randomized algorithms is to split the input into buckets (randomly), solve the problem inside each bucket using some naive algorithm, and then combine the results somehow. So, consider the situation when the input is made out of $n$ items, and $k$ buckets. Each item is being thrown into one of the buckets uniformly (and independently) at random.

   For all the elements in a single bucket, we apply an algorithm that takes $O(n_i^3)$ time, where $n_i$ are the number of elements in the bucket. What is the total expected running time of this stage of the algorithm (over all buckets)?

3. Good cycles. [**10 Points**]

   Prove the following claim.

Let $G = (V, E)$ be a directed graph with outdegree $\delta$ for all vertices, and maximum indegree $\Delta$. Let $k$ be a positive integer such that $e(\Delta\delta + 1)\left(1 - \frac{1}{k}\right)^{\delta} < 1$, then there exists a (directed) cycle $\pi$ in $G$, such that the length of $\pi$ is a multiple of $k$.

(Hint: Randomly color the vertices of the graph by $0, \ldots, k-1$, and let $f$ be the coloring. Argue, that by using the Lovasz local lemma, there exists a coloring, such that for each vertex $v$, there exists a neighbor $u$, such that $f(u) \equiv (f(v) + 1) \bmod k$. Use this to prove the claim.)

4. Independence in a cycle.
**[10 Points]**

Let $G = (V, E)$ be a cycle of length $4n$ and let $V = V_1 \cup V_2 \cup \cdots \cup V_n$ be a partition of its $4n$ vertices into $n$ pairwise disjoint subsets, each of cardinality 4. Is it true that there must be an independent set of $G$ containing precisely one vertex from each $V_i$? (Prove, or supply a counter example.)

5. Random walk. **[10 Points]**

Consider the random walk on the integer numbers starting at 0. Let $\alpha_n$ be the expected number of steps for this walk to reach either $-n$ or $+n$. Give exact expression for $\alpha_n$. (Hint: Initially you can assume $n$ is a power of 2.)