This is a closed-book, closed-notes, open-brain exam. If you brought anything with you besides writing instruments and your handwritten $8\frac{1}{2}'' \times 11''$ cheat sheet, please leave it at the front of the classroom.

Print your name, netid, and alias in the boxes above. Circle $U$ if you are an undergrad, or $G$ if you are a grad student. Print your name at the top of every page (in case the staple falls out!).

You should answer all the questions on the exam.

The last few pages of this booklet are blank. Use that for a scratch paper. Please let us know if you need more paper.

If your cheat sheet if not hand written by yourself, or it is photocopied, please do not use it and leave it in front of the classroom.

Please submit your cheat sheet together with your exam. An exam without your cheat sheet attached to it will not be checked.

If you are NOT using a cheat sheet you should indicate it in large friendly letters on this page.

Questions containing the expression: “I don't know”, will get 25% of the points of the question. If you write anything else, it would be ignored.

The total number of points given for “I don't know” answers, will not exceed 10.

Write short and concise answers. Long and tedious answers will not be graded and will get grade zero automatically.

Time limit: 170 minutes.

Relax. The semester is over.

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1. **Sweeping**  
   [20 Points]

   (A) [10 Points] Given two \(x\)-monotone polygons \(P\) and \(Q\), show how to compute their intersection polygon (which might be made out of several connected components) in \(O(n)\) time, where \(n\) is the total number of vertices of \(P\) and \(Q\). (300 words)\(^1\)

   (B) [10 Points] You are given a set \(\mathcal{H} = \{h_1, \ldots, h_n\}\) of \(n\) half-planes (a half-plane is the region defined by a line - it is either all the points above a given line, or below it). Using (A), show an algorithm to compute the convex polygon \(\cap_{i=1}^{n} h_i\) in \(O(n \log n)\) time. (60 words)

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\(^1\)Your solution for this subquestion should not exceed 300 words. One line of text is about ten words. Note that the limit is quite conservative. Much shorter answers (that would get full credit) are possible and would cause us infinite happiness.
2. **Vertex Cover**  
[20 Points]

**Problem:** VERTEX COVER

**Instance:** A graph $G = (V, E)$ and a positive integer $K \leq |V|$.

**Question:** Is there a vertex cover of size $K$ or less for $G$, that is, a subset $V' \subseteq V$ such that $|V'| \leq K$ and for each edge $\{u, v\} \in E$, at least one of $u$ and $v$ belongs to $V'$?

(A) **[9 Points]** Prove that VERTEX COVER is NP-Complete. (200 words)

(B) **[6 Points]** Show a polynomial approximation algorithm to the VERTEX-COVER problem which is a factor 2 approximation of the optimal solution. Namely, your algorithm should output a set $X \subseteq V$, such that $X$ is a vertex cover, and $|C| \leq 2K_{opt}$, where $K_{opt}$ is the cardinality of the smallest vertex cover of $G$. (100 words)

(C) **[5 Points]** Prove that your approximation algorithm from (B) indeed provides a factor 2 approximation. (30 words)
3. **Majority Tree**  
**[20 Points]**

Consider a uniform rooted tree of height $h$ (every leaf is at distance $h$ from the root). The root, as well as any internal node, has 3 children. Each leaf has a boolean value associated with it. Each internal node returns the value returned by the majority of its children. The evaluation problem consists of determining the value of the root; at each step, an algorithm can choose one leaf whose value it wishes to read.

A. **[5 Points]** Describe a deterministic algorithm that runs in $O(n)$ time, that computes the value of the tree, where $n = 3^h$.

B. **[5 Points]** Describe (i.e., provide pseudo-code) a randomized algorithm for this problem, which is faster than the deterministic algorithm.

C. **[10 Points]** Prove that the expected number of leaves read by your randomized algorithm on any instance is at most $O(n^c)$ (modify your randomized algorithm to achieve this if necessary), where $c$ is a constant smaller than 1. (Of course, no credit would be given to an algorithm with expected linear running time.) (160 words)
4. **Union Find**
   
   [20 Points]

   In the following, we consider a union-find data-structure constructed for $n$ elements.

   A. **[5 Points]** For an element $x$ in the union-find data-structure, describe how $\text{rank}(x)$ is being computed. (100 words)

   B. **[5 Points]** Prove that during the execution of union-find there are at most $n/2^k$ elements that are assigned rank $k$. (100 words)

   C. **[5 Points]** Prove that in a set of $n$ elements, a sequence of $n$ consecutive $\text{FIND}$ operations take $O(n)$ time in total. (100 words)

   D. **[5 Points]** Prove that in the worst case, the time to perform a single $\text{FIND}$ operation is $O(\log n)$. (100 words)
5. **Add Them Up**

**[20 Points]**

Consider two sets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_m\}$, each having at most $n$ integers in the range from 0 to $10n$. We wish to compute the *Cartesian sum* of $A$ and $B$, defined by

$$ C = \{x + y \mid x \in A \text{ and } y \in B\}. $$

Note that the integers in $C$ are in the range from 0 to $20n$. We want to find the elements of $C$ and the number of times each element of $C$ is realized as a sum of elements in $A$ and $B$.

A. **[10 Points]** Show how to reduce this problem to the problem of polynomial multiplication. (70 words)

B. **[10 Points]** Present an algorithm that solves this problem in $O(n \log n)$ time. (Partial credit would be given for a subquadratic algorithm for this problem. Slower algorithms would not get any points.) (60 words)