• This is a closed-book, closed-notes, open-brain exam. If you brought anything with you besides writing instruments and your handwritten \(8\frac{1}{2}'' \times 11''\) cheat sheet, please leave it at the front of the classroom.

• Print your name, netid, and alias in the boxes above. Circle \(U\) if you are an undergrad, or \(G\) if you are a grad student. Print your name at the top of every page (in case the staple falls out!).

• **You should answer all the questions on the exam.**

• The last few pages of this booklet are blank. Use that for a scratch paper. Please let us know if you need more paper.

• If your cheat sheet is not hand written by yourself, or it is photocopied, please do not use it and leave it in front of the classroom.

• Please submit your cheat sheet together with your exam. An exam without your cheat sheet attached to it will not be checked.

• If you are NOT using a cheat sheet you should indicate it in large friendly letters on this page.

• The total number of points given for “I don’t know” answers, will not exceed 10.

• **Write short and concise answers. Long and tedious answers will not be graded and will get grade zero automatically.**

• Time limit: 75 minutes.

• Relax. The semester is almost over...

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1. **MATCHINGS**

[25 Points]

Let be a matching $M$ in an undirected graph $G$, such that $M$ is not a maximum matching. A **fixup path** is a path $v_1, v_2, \ldots, v_k$ in $G$, such that $v_1$ and $v_k$ are unmatched by $M$ (i.e., there is no edge in $M$ adjacent to $v_1$ and $v_k$), the edges $v_2v_3, v_4v_5, v_6v_7, \ldots, v_{k-2}v_{k-1}$ are in $M$, and $v_1v_2, v_3v_4, v_5v_6, \ldots, v_{k-1}v_k$ are **not** in $M$.

For example, in the following graph:

![Graph](image)

$v_1v_2v_3v_4$ is a fixup path.

**A. [10 Points]** Given a matching $M$, and a fixup path $\pi$ for $M$, describe an algorithm that computes a matching $M'$ such that $|M'| > |M|$.

**B. [15 Points]** Prove that for a non-maximum matching $M$ in a graph $G$, there always exists a fixup path for $M$.

(You can not assume that $G$ is bipartite. However, partial credit would be given for a proof that works only for the special case that $G$ is bipartite.)
2. Sort those Numbers
[25 Points]

You had decided to build a sorting network, and you bought enough gates from your supplier Cheap Gates. Unfortunately, instead of the high quality comparators you expected, you got random comparators. Formally, a random comparator, receives two inputs, and with probability half, do nothing (i.e., passes the inputs directly to the outputs), and with probability half, it works correctly outputting the maximum number on the max output, and the minimum on the min output.

Describe a construction of a sorting network, that uses only random comparators, and sort correctly the $n$ inputs, with probability $\geq 1 - 1/n$. How many gates does your sorting network have? Provide a proof that your sorting network works with this required confidence.
3. **Almost Magic Square**

   **[25 Points]**

   You are asked to fill the entries of an $n \times n$ matrix $A$ by integers between 0 and a bound $k$, so that the sum of all entries in each row, and each column, comes to one of $2n$ numbers prespecified in advance. For example, the following instance

   $\begin{array}{ccc}
   17 & 5 & 4 \\
   6 & \text{(? ? ?)} \\
   9 & \text{(? ? ?)} \\
   11 & \text{(? ? ?)} \\
   \end{array}$

   with $k = 9$ has a solution

   $\begin{array}{ccc}
   17 & 5 & 4 \\
   6 & 0 & 0 \\
   9 & 3 & 4 \\
   11 & 2 & 0 \\
   \end{array}$

   Assume that the sum of the rows be specified in an array $R[1..n]$, and the sum of the columns specified in an array $C[1..n]$.

   **A. [15 Points]** Formulate this problem as a network flow problem.
   
   **B. [10 Points]** Write an algorithm for this problem and analyze its running time.
4. Numbers.  

[25 Points]  

Prove that if $p$ is prime, then $(p - 1)! \equiv -1 \pmod{p}$. 