Neatly print your name (first name first, with no comma), your network ID, and a short alias into the boxes above. **Do not sign your name. Do not write your Social Security number.** Staple this sheet of paper to the top of your homework.

Grades will be listed on the course web site by alias give us, so your alias should not resemble your name or your Net ID. If you don’t give yourself an alias, we’ll give you one that you won’t like.

This homework tests your familiarity with the prerequisite material from CS 173, CS 225, and CS 273—many of these problems have appeared on homeworks or exams in those classes—primarily to help you identify gaps in your knowledge. **You are responsible for filling those gaps on your own.** Parberry and Chapters 1–6 of CLR should be sufficient review, but you may want to consult other texts as well.

Before you do anything else, read the Homework Instructions and FAQ on the CS 373g course web page ([http://www-courses.cs.uiuc.edu/~cs373g/hw/faq.html](http://www-courses.cs.uiuc.edu/~cs373g/hw/faq.html)), and then check the box below. This web page gives instructions on how to write and submit homeworks—staple your solutions together in order, write your name and netID on every page, don’t turn in source code, analyze everything, use good English and good logic, and so forth.

☐ I have read the CS 373g Homework Instructions and FAQ.
Required Problems

“I don’t know why it should be, I am sure; but the sight of another man asleep in bed when I am up, maddens me. It seems to me so shocking to see the precious hours of a man’s life - the priceless moments that will never come back to him again - being wasted in mere brutish sleep.”

– Three men in a boat, Jerome K. Jerome

1. **Cornification**

   [20 Points]

   During the sweetcorn festival in Urbana, you had been kidnapped by an extreme anti-corn organization called Al Corona. To punish you, they give you several sacks with a total of \((n + 1)n/2\) cobs of corn in them, and an infinite supply of empty sacks. Next, they ask you to play the following game: At every point in time, you take a cob from every non-empty sack, and you put this set of cobs into a new sack. The game terminates when you have \(n\) non-empty sacks, with the \(i\)th sack having \(i\) cobs in it, for \(i = 1, \ldots, n\).

   For example, if we started with \(\{1, 5\}\) (i.e., one sack has 1 cob, the other 5), we would have the following sequence of steps: \(\{2, 4\}, \{1, 2, 3\}\) and the game ends.

   (a) [5 Points] Prove that the game terminates if you start from a configuration where all the cobs are in a single sack.

   (b) [5 Points] Provide a bound, as tight as possible, on the number of steps in the game till it terminates in the case where you start with a single sack.

   (c) [5 Points] (hard) Prove that the game terminates if you start from an arbitrary configuration where the cobs might be in several sacks.

   (d) [5 Points] Provide a bound, as tight as possible, on the number of steps in the game till it terminates in the general case.

2. **Recurrences**

   [20 Points]

   Solve the following recurrences. State tight asymptotic bounds for each function in the form \(\Theta(f(n))\) for some recognizable function \(f(n)\). You do not need to turn in proofs (in fact, please don’t turn in proofs), but you should do them anyway just for practice. Assume reasonable but nontrivial base cases if none are supplied. More exact solutions are better.

   (a) [1 Points] \(A(n) = A(n/3 + 5 + \lfloor \log n \rfloor) + n \log \log n\)

   (b) [1 Points] \(B(n) = \min_{0 < k < n} (3 + B(k) + B(n - k))\).

   (c) [1 Points] \(C(n) = 3C(\lfloor n/2 \rfloor - 5) + n/\log n\)

   (d) [1 Points] \(D(n) = \frac{n}{n-5} D(n - 1) + 1\)

   (e) [1 Points] \(E(n) = E(\lfloor 3n/4 \rfloor) + 1/\sqrt{n}\)

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1 Cornification - Conversion into, or formation of, horn; a becoming like horn. Source: Webster's Revised Unabridged Dictionary
(f) [1 Points] \( F(n) = F(\lceil \log^2 n \rceil) + \log n \)

(g) [1 Points] \( G(n) = n + 7\sqrt{n} \cdot G(\lfloor \sqrt{n} \rfloor) \)

(h) [1 Points] \( H(n) = \log^2(H(n-1)) + 1 \)

(i) [1 Points] \( I(n) = I(\lfloor n^{1/4} \rfloor) + 1 \)

(j) [1 Points] \( J(n) = J(n - \lfloor n/\log n \rfloor) + 1 \)

3. Sorting functions

[20 Points]

Sort the following 25 functions from asymptotically smallest to asymptotically largest, indicating ties if there are any. You do not need to turn in proofs (in fact, please don’t turn in proofs), but you should do them anyway just for practice.

\[
\begin{array}{cccccc}
\text{n}^{5.5} - (n-1)^{5.5} & n & n^{2.2} & \lg^*(n/7) & 1 + \lg \lg n \\
\cos n + 2 & \lg(\lg^* n) & \lg(n!) & (\lg^* n)^{\lg n} & n^4 \\
\lg^* 2^{2^n} & 2^{\lg^* n} & e^{\sqrt{n}} & \sum_{i=1}^{n} \frac{1}{i} & \frac{\sum_{i=1}^{n} \frac{1}{i^2}}{}
\end{array}
\]

To simplify notation, write \( f(n) \ll g(n) \) to mean \( f(n) = o(g(n)) \) and \( f(n) \equiv g(n) \) to mean \( f(n) = \Theta(g(n)) \). For example, the functions \( n^2 \), \( n \), \( \binom{n}{2} \), \( n^3 \) could be sorted either as \( n \ll n^2 \ll (\binom{n}{2}) \ll n^3 \) or as \( n \ll (\binom{n}{2}) \ll n^2 \ll n^3 \).

4. [20 Points] There are \( n \) balls (numbered from 1 to \( n \)) and \( n \) boxes (numbered from 1 to \( n \)). We put each ball in a randomly selected box.

(a) [8 Points] A box may contain more than one ball. Suppose \( X \) is the number on the box that has the smallest number among all nonempty boxes. What is the expectation of \( X \)? (It’s OK to just give a big expression.)

(b) [8 Points] We put the balls into the boxes in such a way that there is exactly one ball in each box. If the number written on a ball is the same as the number written on the box containing the ball, we say there is a match. What is the expected number of matches?

(c) [4 Points] What is the probability that there are exactly \( k \) matches? (1 \( \leq k < n \))

[Hint: If you have to appeal to “intuition” or “common sense”, your answers are probably wrong!]

5. Coloring

[10 Points]
(a) [5 Points] Let $T_1, T_2$ and $T_3$ be three trees defined over the set of vertices \( \{v_1, \ldots, v_n\} \). Prove that the graph $G = T_1 \cup T_2 \cup T_3$ is colorable using six colors (\( e \) is an edge of $G$ if and only if it is an edge in one of trees $T_1$, $T_2$ and $T_3$).

(b) [5 Points] Describe an efficient algorithm for computing this coloring. What is the running time of your algorithm?

6. Random Elections
[10 Points]

You are in a shop trying to buy green tea. There $n$ different types of green tea that you are considering: $T_1, \ldots, T_n$. You have a coin, and you decide to randomly choose one of them using random coin flips. Because of the different prices of the different teas, you decide that you want to choose the $i$th tea with probability $p_i$ (of course, $\sum_{i=1}^{n} p_i = 1$).

Describe an algorithm that chooses a tea according to this distribution, using only coin flips. Compute the expected number of coin flips your algorithm uses. (Your algorithm should minimize the number of coin flips it uses, since if you flip coins too many times in the shop, you might be arrested.)