Neatly print your name(s), NetID(s), and the alias(es) you used for Homework 0 in the boxes above. Please also tell us whether you are an undergraduate or a grad student by circling U or G, respectively. Staple this sheet to the top of your homework.

**Note:** You will be held accountable for the appropriate responses for answers (e.g. give models, proofs, analysis, etc). For NP-complete problems you should prove everything rigorously, i.e. for showing that it is in NP, give a description of a certificate and a polynomial time algorithm to verify it, and for showing NP-hardness, you must show that your reduction is polynomial time (by similarly proving something about the algorithm that does the transformation) and proving both directions of the ‘if and only if’ (a solution of one is a solution of the other) of the many-one reduction.
Only of myself I know how to tell,  
my world is as narrow as an ant’s.  
like an ant too my burden I carry,  
too great and heavy for my frail shoulder.

My way too - like the ant’s to the treetop -  
is a way of pain and toil;  
a gigantic hand, assured and malicious,  
a mocking hand hinders

All my paths are made bleak and tearful  
by the constant dread of this giant hand.

Why do you call to me, wondrous shores?  
Why do you lie to me, distant lights?

– Only of Myself, Rachel

Required Problems

1. Beware of Greeks bearing gifts\(^1\)

[20 Points]

The reduction faun, the brother of the Partition satyr, came to visit you on labor day, and left you with two black boxes.

(a) [10 Points] The first black box, was a black box that can solves the following decision problem in polynomial time:

**Problem:** MINIMUM TEST COLLECTION

| Instance: A finite set \( A \) of “possible diagnoses,” a collection \( C \) of subsets of \( A \), representing binary “tests,” and a positive integer \( J \leq |C| \).
| Question: Is there a subcollection \( C' \subseteq C \) with \( |C'| \leq J \) such that, for every pair \( a_i, a_j \) of possible diagnoses from \( A \), there is some test \( c \in C' \) for which \( |\{a_i, a_j\} \cap c| = 1 \) (that is, a test \( c \) that “distinguishes” between \( a_i \) and \( a_j \) )?

Show how to use this black box, how to solve in polynomial time the optimization version of this problem (i.e., finding and outputting the smallest possible set \( C' \)).

(b) [10 Points]  
The second box was a black box for solving

**Subgraph Isomorphism.**

**Problem:** SUBGRAPH ISOMORPHISM

| Instance: Two graphs, \( G = (V_1, E_1) \) and \( H = (V_2, E_2) \).
| Question: Does \( G \) contain a subgraph isomorphic to \( H \), that is, a subset \( V \subseteq V_1 \) and a subset \( E \subseteq E_1 \) such that \( |V| = |V_2|, |e| = |E_2| \), and there exists a one-to-one function \( f : V_2 \to V \) satisfying \( \{u, v\} \in E_2 \) if and only if \( \{f(u), f(v)\} \in E \)?

Show how to use this black box, to compute the subgraph isomorphism (i.e., you are given \( G \) and \( H \), and you have to output \( f \)) in polynomial time.

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\(^1\)Based on Virgil’s Aeneid: “Quidquid id est, timeo Danaos et dona ferentes”, which means literally “Whatever it is, I fear Greeks even when they bring gifts.”
2. Graph Isomorphisms [10 Points]

(a) [5 Points] Show that the following problem is NP-Complete.

**Problem:** SUBGRAPH ISOMORPHISM

**Instance:** Graphs \( G = (V_1, E_1) \), \( H = (V_2, E_2) \).

**Question:** Does \( G \) contain a subgraph isomorphic to \( H \), i.e., a subset \( V \subseteq V_1 \) and a subset \( E \subseteq E_1 \) such that \(|V| = |V_2|, |E| = |E_2|\), and there exists a one-to-one function \( f : V_2 \to V \) satisfying \( \{u, v\} \in E_2 \) if and only if \( \{f(u), f(v)\} \in E \)?

(b) [5 Points] Show that the following problem is NP-Complete.

**Problem:** LARGEST COMMON SUBGRAPH

**Instance:** Graphs \( G = (V_1, E_1) \), \( H = (V_2, E_2) \), positive integer \( K \).

**Question:** Do there exists subsets \( E'_1 \subseteq E_1 \) and \( E'_2 \subseteq E_2 \) with \(|E'_1| = |E'_2| \geq K \) such that the two subgraphs \( G' = (V_1, E'_1) \) and \( H' = (V_2, E'_2) \) are isomorphic?

3. Subset Sum [20 Points]

**Problem:** SUBSET SUM

**Instance:** \( S \) - set of positive integers, \( t \) - an integer number

**Question:** Is there a subset \( X \subseteq S \) such that

\[
\sum_{x \in X} x = t ?
\]

Given an instance of SUBSET SUM, provide an algorithm that solves it in polynomial time in \( n \), and \( M \), where \( M = \max_{s \in S} s \). Why this does not imply that \( P = NP \)?

4. 2SAT [10 Points]

Given an instance of 2SAT (this is a problem similar to 3SAT where every clause has at most two variables), one can try to solve it by backtracking.

(a) [1 Points] Prove that if a formula \( F' \) is not satisfiable, and \( F \) is formed by adding clauses to \( F' \), then the formula \( F \) is not satisfiable. (Duh?)

We refer to \( F' \) as a subformula of \( F \).

(b) [3 Points] Given an assignment \( x_i \leftarrow b \) to one of the variables of a 2SAT instance \( F \) (where \( b \) is either 0 or 1), describe a polynomial time algorithm that computes a subformula \( F' \) of \( F \), such that (i) \( F' \) does not have the variable \( x_i \) in it, (ii) \( F' \) is a 2SAT formula, (iii) \( F' \) is satisfiable iff there is a satisfying assignment for \( F \) with \( x_i = b \), and (iv) \( F' \) is a subformula of \( F \).

How fast is your algorithm?

(c) [6 Points] Describe a polynomial time algorithm that solves the 2SAT problem (using (b)). How fast is your algorithm?
5. **Hamiltonian Cycle Revisited** [20 Points]

Let \( C_n \) denote the cycle graph over \( n \) vertices (i.e., \( V(C_n) = \{1, \ldots, n\} \) and \( E(C_n) = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\} \)). Let \( C^k_n \) denote the graph where \( \{i, j\} \in E(C^k_n) \) iff \( i \) and \( j \) are in distance at most \( k \) in \( C_n \).

Let \( G \) be a graph, such that \( G \) is a subgraph of \( C^k_n \), where \( k \) is a small constant. Describe a polynomial time algorithm (in \( n \)) that outputs a Hamiltonian cycle if such a cycle exists in \( G \). How fast is your algorithm, as a function of \( n \) and \( k \)?

6. **NP Completeness** [20 Points]

(a) [5 Points]
**Problem:** MINIMUM SET COVER

**Instance:** Collection \( C \) of subsets of a finite set \( S \) and an integer \( k \).

**Question:** Are there \( k \) sets \( S_1, \ldots, S_k \) in \( C \) such that \( S \subseteq \bigcup_{i=1}^{k} S_i \)?

(b) [5 Points]
**Problem:** BIN PACKING

**Instance:** Finite set \( U \) of items, a size \( s(u) \in \mathbb{Z}^+ \) for each \( u \in U \), an integer bin capacity \( B \), and a positive integer \( K \).

**Question:** Is there a partition of \( U \) into \( K \) disjoint sets \( U_1, \ldots, U_K \) such that the sum of the sizes of the items inside each \( U_i \) is \( B \) or less?

(c) [5 Points]
**Problem:** TILING

**Instance:** Finite set \( R \) of rectangles and a rectangle \( R \) in the plane.

**Question:** Is there a way of placing the rectangles of \( R \) inside \( R \), so that no pair of the rectangles intersect, and all the rectangles have their edges parallel of the edges of \( R \)?

(d) [5 Points]
**Problem:** HITTING SET

**Instance:** A collection \( C \) of subsets of a set \( S \), a positive integer \( K \).

**Question:** Does \( S \) contain a hitting set for \( C \) of size \( K \) or less, that is, a subset \( S' \subseteq S \) with \( |S'| \leq K \) and such that \( S' \) contains at least one element from each subset in \( C \).