1 Combinatorics & Discrete Probability

Example 1.1 How many word Scrabbles can we make from ROSEN?
  SNORE, NOSER, SENOR, ...

\[
\begin{array}{cccccc}
\& & \& & \& & \\
\end{array}
\]

\[
5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!
\]

by Rule of product.
We will consider 0! = 1.

Example 1.2 Q: How many different bijective functions are there between \(X\) to \(X\), where \(|X| = 5\)?
  Draw a function.
  Define onto function,
  define one to one function
  Define bijective function = one to one + onto.

\[
\begin{array}{cccccc}
R \rightarrow S \\
O \rightarrow N \\
S \rightarrow O \\
E \rightarrow R \\
N \rightarrow E
\end{array}
\]

Draw graph showing function.

Thus, a bijective function from \(X\) to \(X\) is just an ordering of \(X\). This is known as a permutation of \(X\).
Theorem 1.3 There are \( n! \) ways to order a collection of \( n \) objects.
(Namely, there \( n! \) permutations of \( n \) objects.)

What if elements are indistinguishable?

Example 1.4 Word scrabble: FREEDOM. How many different ways to order it?
If \( FRE_1E_2DOM \) then \( 7! \).

Example 1.5 Word scrabble: MISSISSIPPI. How many different ways to order it?

Theorem 1.6 There are \( \frac{n!}{n_1!n_2!\cdots n_k!} \) ways to order \( n \) elements with \( k \) groups of identical elements with sizes \( n_1, n_2, \ldots, n_k \).

Proof: (By induction.)
Base case: \( k = 1 \), all characters are identical, and there are only \( n!/n! = 1 \) way to do it.
Induction step: Assume true for \( k = i - 1 \), show for \( k = i \). So assume the \( i \)th group elements are indistinguishable, then there are

\[
\frac{n!}{n_1!\cdots n_{k-1}!}
\]

ways. Now, if the elements of the \( i \)th groups are not distinguishable, then we are counting every distinct ordering \( n_k! \) times, by the overcounting argument. Thus, there are

\[
\frac{n!}{n_1!\cdots n_{k-1}!} \cdot \frac{1}{n_k!}
\]

ways, as claimed.

What if we consider ordering of \( r \) objects, but we have \( n \geq r \) to choose from?

Example 1.7 Using the English alphabet, how many 4-letter words can be made, assuming there are no words with two identical letters?

Theorem 1.8 There are \( P(n, r) = n(n-1) \cdots (n-r+1) \) ways to make an ordered lists of \( r \) objects from a collection \( n \) different objects (without replacement).

Note that \( P(n, r) = \)
Such orderings are sometime called \( r \)-permutations.
Here, whenever we choose an object, we “take it away” and it can not be used again. This is known as choosing \textit{without replacement}. 
Q: What if allow replacement?

**Theorem 1.9** There are \( n^r \) ways to make an ordered list of \( r \) objects from a collection of \( n \) different objects with replacement allowed.

**Example 1.10** How many strings of length \( r \) can be formed from \( n \) symbols?

\[ n^r \]

**Example 1.11** How many functions \( f : X \rightarrow Y \) such that \( |X| = r \) and \( |Y| = n \)?

\[ n^r \]

Now return to the case of without replacement, but let’s forget about ordering.

**Example 1.12** How many ways can we choose a set of 4 letters from the letters of the English alphabet?

If we can about ordering, then \( P(26, 4) = \frac{26!}{24!} \) \( r \)-permutations. Note, that we have the word

\[ BDGX \]

a lot of times with different order:

\[ BGDX, BXGD, XGDB... \]

How many exactly?

\[ 4! \]

As such, the number of such sets is

\[ \frac{26!}{22!4!} = \frac{26 \cdot 25 \cdot 24 \cdot 23}{4!} = \frac{26 \cdot 25 \cdot 24 \cdot 23}{24} = 25 \cdot 24 \cdot 23. \]

**Theorem 1.13** There are \( \frac{n!}{(n-r)!r!} \) ways to select \( r \) objects from a collection of \( n \) distinct objects (w/o replacement) if order is irrelevant.

Notation:

\[ C(n, r) \equiv \binom{n}{r} \equiv \frac{n!}{(n-r)!r!} \]

We read “\( \binom{n}{r} \)” as “\( n \) choose \( r \)”.

\( \binom{n}{r} \) is the number of \( r \)-combinations of \( n \) elements.

Overall, we have:
• Permutations
• r-permutations
• r-combinations

**Example 1.14** We have \( r \) identical white balls, and \( n - r \) identical black balls, how many different ways are there to order all the balls in a row?

Suppose that we have 10 identical (indistinguishable) balls, and 5 different boxes. How many different ways do we have to partition the balls into the boxes? (several of the boxes might be empty - thats OK)

\[
\begin{array}{llll}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & | & | & \cdot & \cdot \\
\end{array}
= 7, 1, 0, 2, 0
\]

\[
\begin{array}{llllll}
| & \cdot & \cdot & \cdot & | & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
= 0, 3, 3, 4, 0
\]

Each | divides between two consecutive boxes. There are 4 such splitters.

**Observation 1.15** For every configuration of balls and boxes there is a unique sequence of balls and splitters that encodes it.

So, count the number of such sequences. The number of such sequences is

\[
\binom{10 + 4}{4}
\]

Note that this also the result for 11 boxes and 4 balls.

**Theorem 1.16** The number of ways to place \( r \) indistinguishable balls into \( n \) boxes is \( \binom{r + n - 1}{n - 1} \).

**Example 1.17** How many solutions are there to the equation

\[
x_1 + x_2 + \cdots + x_n = r
\]

such that \( x_i \geq 0, \forall i \in \{1, \ldots, n\} \) and \( x_i \) is an integer?