1 Discrete Probability

Probability of an events is always in the range \([0, 1]\).

Tossing coin: Probability of Heads is \(1/2\)

Tossing a die: Probability of 6 is \(1/6\).

Where do probabilities comes from:

1. What are your chances of getting an A in 273?
   What are the chances that one of the daughters of G. W. Bush would be president?

2. Assume all outcomes are equally likely (Laplace)
   Sides of a die, sides of a coins, etc.

3. Do experiments. Count number of successes to the number of trials. This gives you some probability.


Definition 1.1 (Probability space) We a tuple \((S, \mathcal{F}, P)\), where \(S\) is the sample space, \(\mathcal{F}\) is the event space (\(\sigma\)-field) and \(P\) is a probability function.

\[ S = \text{sample space - set of all outcomes of an experiments.} \]

\[ \text{Coin toss: } S = \{H, T\}. \]

\[ \text{Die throw: } S = \{1, 2, 3, 4, 5, 6\}. \]

\[ \text{A sequence of three coin throws: } S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}. \]

\[ \text{Total of 2 die throw: } \{2, 3, \ldots, 12\}. \]

\[ \mathcal{F} = \text{Event space - All possible events we might be interested in. Namely, } \mathcal{F} = \text{pow}(S) = 2^S = \{T \mid T \subseteq S\}. - \text{all possible subsets of } S. \]

(Note, that \text{pow}(S) is naturally defined for finite sets. For infinite sets, this becomes a total mess (i.e., set theory)).

Example 1.2 For a single coin flip: \(\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}.\)
• $\Pr[E] \geq 0, \forall E \in \mathcal{F}$.
• $\Pr[S] = 1$
• $\Pr[E \cup F] = \Pr[E] + \Pr[F] = \Pr[E] + \Pr[F]$ if $E \cap F = \emptyset, \forall E, F \in \mathcal{F}$.

Note that for every $x \in S$ we have to define $\Pr[\{x\}]$ because $\{x\} \in \mathcal{F}$.

From the rules of probability:

$$\sum_{x \in S} \Pr[\{x\}] = 1$$

and $\Pr[\{x\}] \geq 0$ for all $x \in S$.

Thus, probability is like a mass (of size 1) that is smeared around among the elements of $S$. Every element has its own probability.

**Example 1.3**

$\Pr[\{a\}] = 0.25 \Pr[\{b\}] = 0.25 \Pr[\{c\}] = 0.125 \Pr[\{d\}] = 0.375.$

$\Pr[\{a, d\}] = 0.25 + 0.375 = 0.625.$

**Example 1.4** Suppose we toss a coin twice. The outcomes are $HH, HT, TH, TT$ and thus $S = \{HH, HT, TH, TT\}$ and $\Pr[HH] = 1/4$, $\Pr[HT] = 1/4$, $\Pr[TH] = 1/4$, and $\Pr[TT] = 1/4$.

What about? $\Pr[\{HH, HT\}] = 0.5$ and $\Pr[\{HH, HT, TT\}] = 0.75$.

If the probability mass is “smeared” evenly, we obtain nice counting problems.

Suppose $\Pr[\{x\}] = \frac{1}{|S|}, \forall x \in S$.

**Example 1.5** What is the probability that a hand of $S$ cards contains exactly 3 of a kind?

• Assume all hands are equally likely.
• Probability of each hand is

$$\Pr[\{x\}] = \frac{1}{|S|},$$

where $|S| = \binom{52}{5}$.

Thus, the number of such hands is $13 \binom{4}{3} \binom{48}{2}$, and

$$\Pr[\text{A hand of } S \text{ contains 3 of a kind}] = \frac{13 \binom{4}{3} \binom{48}{2}}{\binom{52}{5}}.$$

If $E \cap F = \emptyset$ then $\Pr[E \cup F] = \Pr[E] + \Pr[F]$. How to compute $\Pr[E \cup F]$ if $E \cap F \neq \emptyset$?

Well, let’s use inclusion/exclusion:

$$\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F].$$

This is very similar to counting, when the elements have different weights (i.e., their probability). Similarly, we can compute

$$\Pr[E_1 \cup E_2 \cup \cdots \cup E_m] = ?$$
using the inclusion/exclusion principle.

\[ \Pr[\overline{E}] = \Pr[S] - \Pr[E] = 1 - \Pr[E]. \]

One natural way of thinking of events, is that they all the combinations you can create using complement, union, of conjunction.

1.1 Conditional Probability

Example 1.6 What is the probability of getting 3 in a die throw, if we know that the die result was an odd number?

\[ \frac{|\{3\}|}{|\{1, 3, 5\}|}. \]

Definition 1.7 (Conditional Probability) Given two events \( A \) and \( B \), we denote by \( \Pr[A | B] \) the conditional probability of \( A \) given \( B \).

To compute \( \Pr[A | B] \), we realize that this is the probability of \( A \cap B \) to happened, when we know that \( B \) happened. Namely,

\[ \Pr[A | B] = \frac{\Pr[A \cap B]}{\Pr[B]}. \]

Example 1.8 \( S \) - set of 100 people in class. Pick one person.

What is the probability of picking \( \{JD\} \)? Let denote this event by \( A \). Let \( B \) be the event of picking a person from the first row (let assume that there are 10 such persons).

Now, what is the probability of \( \Pr[A | B] \)?

Well, if \( JD \) does not sitting in the first row, this is 0, because \( \Pr[A \cap B] = 0 \). Otherwise,

\[ \Pr[A | B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{1/250}{1/25} = 1/10. \]

Be very careful applying conditional probabilities!

Example 1.9 Lets make a deal. You have three doors, behind one door there is a new car, behind one there is a goat, and the third one hides a year supply of class notes for 273.

A car is randomly assigned to a door. Let \( S = \{1, 2, 3\} \) be the events where the car is. Pick a door, w.l.o.g, assume it is door 1.

Next, I show you that either behind door 2, or door 3, there is no car.

Question: Do you want to change your choice?

- Best to stay with current choice?
- Best to change?
- Does not make a difference.

Probability of winning if you do not change your guess is 1/3.

Probability of winning if you do chance your guess is 2/3, because with probability 2/3 the prize is not behind the door you started with, and then if you change your door, you win.