Once I sat on the steps by a gate of David’s Tower, I placed my two heavy baskets at my side. A group of tourists was standing around their guide and I became their target marker. “You see that man with the baskets? Just right of his head there’s an arch from the Roman period. Just right of his head.”

“But he’s moving, he’s moving!”

I said to myself: redemption will come only if their guide tells them, “You see that arch from the Roman period? It’s not important: but next to it, left and down a bit, there sits a man who’s bought fruit and vegetables for his family.”

–Yehuda Amichai, Tourists

1 Preliminaries

Definition 1.1 Given a set of hyperplanes $H$ in $\mathbb{R}^d$, the minimization diagram of $H$, known as the lower envelope of $H$, is the function $L_H : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$, where we have $L(x) = \min_{h \in H} h(x)$, for $x \in \mathbb{R}^{d-1}$.

Similarly, the upper envelope of $H$ is the function $U(x) = \max_{h \in H} h(x)$, for $x \in \mathbb{R}^{d-1}$.

The extent of $H$ and $x \in \mathbb{R}^{d-1}$ is the vertical distance between the upper and lower envelope at $x$; namely, $E_H(x) = U(x) - L(x)$.

2 Motivation - Maintaining the Bounding Box of Moving Points

Let $P = \{p_1, \ldots, p_n\}$ be a set of $n$ points moving in $\mathbb{R}^d$. For a given time $t$, let $p_i(t) = (x_1^i(t), \ldots, x_d^i(t))$ denote the position of $p_i$ at time $t$. We will use $P(t)$ denote the set $P$ at time $t$. We say that the motion of $P$ has degree $k$ if every $x_i^j(t)$ is a polynomial of degree at most $k$. We call a motion of degree 1 linear. Namely, $p_i(t) = a_i + b_i t$, where $a_i, b_i \in \mathbb{R}^d$. The values $a_i, b_i$ are fixed.

Our purpose is to develop efficient approaches for maintaining various descriptors of the extent of $P$, including the smallest enclosing orthogonal rectangle of $P$. This measure indicates how spread out the point set $P$ is. As the points move continuously, the extent measure of interest changes continuously as well, though its combinatorial realization changes only at certain discrete times. For example, the smallest orthogonal rectangle containing $P$ can be represented by a sequence of $2d$ points, each lying on one of the facets of the rectangle. As the points move, the rectangle also changes continuously. At certain discrete times, the points lying on the boundary of the rectangle change, and we have to update the sequence of points defining the rectangle. Similarly, Our
A hyperplane $h : x_d = b_1 x_1 + \cdots + b_{d-1} x_{d-1} - b_d$ in $\mathbb{R}^d$ can be interpreted as a function from $\mathbb{R}^{d-1}$ to $\mathbb{R}$. Given a point $(y_1, \ldots, y_d)$ let $h(y) = b_1 y_1 + \cdots + b_{d-1} y_{d-1} - b_d$. In particular, a point $y$ lies above the hyperplane $h$ if $y_d > h(y)$. Similarly, $y$ is below the hyperplane $h$ if $y_d < h(y)$. Finally, a point is on the hyperplane if $h(y) = y_d$.

The dual of a point $b = (b_1, \ldots, b_d) \in \mathbb{R}^d$ is a hyperplane $b^* : x_d = b_1 x_1 + \cdots + b_{d-1} x_{d-1} - b_d$, and the dual of a hyperplane $h : x_d = a_1 x_1 + a_2 x_2 + \cdots + a_{d-1} x_{d-1} + a_d$ is the point $h^* = (a_1, \ldots, a_{d-1}, -a_d)$. There are several alternative definitions of duality, but they are essentially
For a point $x$, the vertical distance $\omega$ is defined by:

$$b = (b_1, \ldots, b_d) \quad \Rightarrow \quad b^* : x_d = b_1 x_1 + \cdots + b_{d-1} x_{d-1} - b_d$$

And for a set of hyperplanes $H$ and a point $x$, the vertical segment $\omega$ is defined by:

$$h : x_d = a_1 x_1 + a_2 x_2 + \cdots + a_{d-1} x_{d-1} + a_d \quad \Rightarrow \quad h^* = (a_1, \ldots, a_{d-1}, -a_d).$$

The proof of the following lemma is straightforward, and is delegated to Exercise 9.1.

**Lemma 3.1** For a point $b = (b_1, \ldots, b_d)$, we have:

(i) $b^* = b$.

(ii) The point $b$ lies above (resp. below, on) the hyperplane $h$, if and only if the point $h^*$ lies above (resp. below, on) the hyperplane $b^*$.

(iii) The vertical distance between $b$ and $h$ is the same as that between $b^*$ and $h^*$.

(iv) The vertical distance $\delta(h, g)$ between two parallel hyperplanes $h$ and $g$ is the same as the length of the vertical segment $h^* g^*$.

(v) For $x \in \mathbb{R}^{d-1}$, the hyperplane $h$ dual to the point $L_H(x)$ (resp. $U_H(x)$) is normal to the vector $(x, -1) \in \mathbb{R}^d$ and supports $CH(H^*)$. Furthermore, $H^*$ lies below (resp. above) the hyperplane $h$.

Furthermore, $E_H(x)$ is the vertical distance between these two parallel supporting planes.

**Definition 3.2** For a set of hyperplanes $H$, a subset $S \subseteq H$ is an $\varepsilon$-coreset of $H$ if for the extent measure, if for any $x \in \mathbb{R}^{d-1}$ we have $E_S \geq (1 - \varepsilon) E_H$.

Similarly, for a point-set $P \subseteq \mathbb{R}^d$, a set $S \subseteq P$ is an $\varepsilon$-coreset for vertical extent of $P$, if, for any direction $v \in S^{d-1}$, we have that $\mu_v(S) \geq (1 - \varepsilon) \mu_v(P)$, where $\mu_v(P)$ is the vertical distance between the two supporting hyperplanes of $P$ which are perpendicular to $v$.

Thus, to compute a coreset for a set of hyperplanes, it is by duality and Lemma 3.1 enough to find a coreset for the vertical extent of a point-set.

**Lemma 3.3** The set $S$ is an $\varepsilon$-coreset of the point set $P \subseteq \mathbb{R}^d$ for vertical extent if and only if $S$ is an $\varepsilon$-coreset for directional width.

**Proof:** Consider any direction $v \in S^{d-1}$, and let $\alpha$ be its (smaller) angle with with the $x_d$ axis. Clearly, $\omega(v, S) = \mu_v(S) \cos \alpha$ and $\omega(v, P) = \mu_v(PntSet) \cos \alpha$. Thus, if $\omega(v, S) \geq (1 - \varepsilon) \omega(v, P)$ then $\mu_v(S) \geq (1 - \varepsilon) \mu_v(P)$, and vice versa.

**Theorem 3.4** Let $H$ be a set of $n$ hyperplanes in $\mathbb{R}^d$. One can compute a $\varepsilon$-coreset of $H$ of size $O(1/\varepsilon^d)$, in $O(n + \min(n, 1/\varepsilon^d))$ time. Alternatively, one can compute a $\varepsilon$-coreset of size $O(1/\varepsilon^{(d-1)/2})$, in $O(n + 1/\varepsilon^{3(d-1)/2})$ time.

**Proof:** By Lemma 3.3 the coreset computation is equivalent to computing coreset for directional width. However, this can be done in the stated bounds, by Theorem 8.1 and Theorem 8.2.

Going back to our motivation, we have the following result:

**Lemma 3.5** Let $P(t)$ be a set of $n$ points with linear motion in $\mathbb{R}^d$. We can compute an axis parallel moving bounding box $b(t)$ for $P(t)$ that changes $O(d/\sqrt{\varepsilon})$ times (in other times, the bounding box moves with linear motion). The time to compute this bounding box is $O(d(n + 1/\varepsilon^{3/2}))$.

Furthermore, we have that $Box(P(t)) \subseteq b(t) \subseteq (1 + \varepsilon) Box(P(t))$, where $Box(t)$ is the minimum axis parallel bounding box of $P$. 
Given a set $\omega H$ (upper/lower envelopes and extent of any function $f$) interested in the value of $F$ 5 Extent of Polynomials

sufficient to show the existence of coresets for a large family of problems.

Thus, for any point $(x, y)$, we can evaluate $H$ on $\eta(x, y)= (x, y, x^2+y^2)$, where $\eta(x, y)$ is the linearization image of $(x, y)$. The advantage of this linearization is that $H$, being a family of linear functions, is now easier to handle than $F$.

Observe, that $X = \eta(\mathbb{R}^2)$ is a subset of $\mathbb{R}^3$ (this is the “standard” paraboloid), and we are interested in the value of $H$ only on points belonging to $X$. In particular, the set $X$ is not necessarily

4 Coresets

At this point, our discussion exposes a very powerful technique for approximate geometric algorithms: (i) extract small subset that represents that data well (i.e., coreset), and (ii) run some other algorithm on the coreset. To this end, we need a more unified definition of coresets.

Definition 4.1 (Coresets) Given a set $P$ of points (or geometric objects) in $\mathbb{R}^d$, and an objective function $f : 2^{\mathbb{R}^d} \to \mathbb{R}$ (say, $f(P)$ is the width of $P$), a $\varepsilon$-coreset is a subset $S$ of the points of $P$ such that

$$f(S) \geq (1 - \varepsilon)f(P).$$

We will state this fact, by saying that $S$ is a $\varepsilon$-coreset of $P$ for $f(\cdot)$.

If the function $f(\cdot)$ is parameterized, namely $f(Q, v)$, then $S \subseteq P$ is a coreset if

$$\forall v \ f(S, v) \geq (1 - \varepsilon)f(P, v).$$

As a concrete example, for $v$ a unit vector, consider the function $CH(v, P)$ which is the directional width of $P$; namely, it is the length of the projection of $CH(P)$ into the direction of $v$.

Coresets are of interest when they can be computed quickly, and have small size, hopefully of size independent of $n$, the size of the input set $P$. Interestingly, our current techniques are almost sufficient to show the existence of coresets for a large family of problems.

5 Extent of Polynomials

Let $F = \{f_1, \ldots, f_n\}$ be a family of $d$-variate polynomials and let $u_1, \ldots, u_d$ be the variables over which the functions of $F$ are defined. Each $f_i$ corresponds to a surface in $\mathbb{R}^{d+1}$. For example, any $d$-variate linear function can be considered as a hyperplane in $\mathbb{R}^{d+1}$ (and vice versa). The upper/lower envelopes and extent of $F$ can now be defined similar to the hyperplane case.

Each monomial over $u_1, \ldots, u_d$ appearing in $F$ can be mapped to a distinct variable $x_i$. Let $x_1, \ldots, x_s$ be the resulting variables. As such $F$ can be linearized into a set $H = \{h_1, \ldots, h_n\}$ of linear functions over $\mathbb{R}^s$. In particular, $H$ is a set of $n$ hyperplanes in $\mathbb{R}^{s+1}$. Note that the surface induced by $f_i$ in $\mathbb{R}^{d+1}$ corresponds only to a subset of the surface of $h_i$ in $\mathbb{R}^{s+1}$. This technique is called linearization.

For example, consider a family of polynomials $F = \{f_1, \ldots, f_n\}$, where $f_i(x, y) = a_i(x^2 + y^2) + b_i x + c_i y + d_i$, and $a_i, b_i, c_i, d_i \in \mathbb{R}$, for $i = 1, \ldots, n$. This family of polynomials defined over $\mathbb{R}^2$, can be linearized to a family of linear functions defined over $\mathbb{R}^3$, by $h_i(x, y, z) = a_i z + b_i x + c_i y + d_i$, and setting $H = \{h_1, \ldots, h_n\}$. Clearly, $H$ is a set of hyperplanes in $\mathbb{R}^4$, and $f_i(x, y) = h_i(x, y, x^2 + y^2)$. Thus, for any point $(x, y) \in \mathbb{R}^2$, instead of evaluating $F$ on $(x, y)$, we can evaluate $H$ on $\eta(x, y) = (x, y, x^2 + y^2)$, where $\eta(x, y)$ is the linearization image of $(x, y)$. The advantage of this linearization is that $H$, being a family of linear functions, is now easier to handle than $F$.

Proof: We compute the solution for each dimension separately. In each dimension, we compute a coreset of the resulting set of lines in two dimensions, and compute the upper and lower envelope of the coreset. Finally, we expand the upper and lower envelopes appropriately so that the include the original upper and lower envelopes. The bounds on the running time follows from Theorem 3.4.


convex. The set $X$ resulting from the linearization is a semi-algebraic set of constant complexity, and as such basic manipulation operations of $X$ can be performed in constant time.

Note that for each $1 \leq i \leq n$, $f_i(p) = h_i(\eta(p))$ for $p \in \mathbb{R}^d$. As such, if $\mathcal{H}' \subseteq \mathcal{H}$ is a $\varepsilon$-coreset of $\mathcal{H}$ for the extent, then clearly the corresponding subset in $\mathcal{F}$ is a $\varepsilon$-coreset of $\mathcal{F}$ for the extent measure. The following theorem is a restatement of Theorem 3.4 in this settings.

**Theorem 5.1** Given a family of $d$-variate polynomials $\mathcal{F} = \{f_1, \ldots, f_n\}$, and parameter $\varepsilon$, one can compute, in $O(n + 1/\varepsilon^2)$ time, a subset $\mathcal{F}' \subseteq \mathcal{F}$ of $O(1/\varepsilon^2)$ polynomials, such that $\mathcal{F}'$ is a $\varepsilon$-coreset of $\mathcal{F}$ for the extent measure. Here $s$ is the number of different monomials present in the polynomials of $\mathcal{F}$.

Alternatively, one can compute a $\varepsilon$-coreset, of size $O(1/\varepsilon^{s/2})$, in time $O(n + 1/\varepsilon^{3s/2})$.

### 6 Roots of Polynomials

We now consider the problem of approximating the extent a family of square-roots of polynomials. Note, that this is considerably harder than handling polynomials because square-roots of polynomials can not be directly linearized. It turns out, however, that it is enough to $O(\varepsilon^2)$-approximate the extent of the functions inside the roots, and take the root of the resulting approximation.

**Theorem 6.1** Let $\mathcal{F} = \{(f_1)^{1/2}, \ldots, (f_n)^{1/2}\}$ be a family of $k$-variate functions (over $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$), where each $f_i$ is a polynomial that is non-negative for every $x \in \mathbb{R}^k$. Given any $\varepsilon > 0$, we can compute, in $O(n + 1/\varepsilon^2k)$ time, a $\varepsilon$-coreset $\mathcal{G} \subseteq \mathcal{F}$ of size $O(1/\varepsilon^{2k})$, for the measure of the extent. Here $k'$ is the number of different monomials present in the polynomials in $f_1, \ldots, f_n$.

Alternatively, one can compute a set $\mathcal{G}' \subseteq \mathcal{F}$, in $O(n + 1/\varepsilon^{3k'})$ time, that $\varepsilon$-approximates $\mathcal{F}$, so that $|\mathcal{G}'| = O(1/\varepsilon^{k'})$.

**Proof:** Let $\mathcal{F}^2$ denote the family $\{f_1, \ldots, f_n\}$. Using the algorithm of Theorem 5.1, we compute a $\delta'$-coreset $\mathcal{G}^2 \subseteq \mathcal{F}^2$ of $\mathcal{F}^2$, where $\delta' = \varepsilon^2/64$. Let $\mathcal{G} \subseteq \mathcal{F}$ denote the family $\{(f_1)^{1/2} | f_1 \in \mathcal{G}^2\}$.

Consider any point $x \in \mathbb{R}^k$. We have that $\mathcal{E}_{\mathcal{G}^2}(x) \geq (1 - \delta')\mathcal{E}_{\mathcal{F}^2}(x)$, and let $A = \mathcal{L}_{\mathcal{F}^2}(x)$, $B = \mathcal{L}_{\mathcal{G}^2}(x)$, and $b = \mathcal{L}_{\mathcal{F}^2}(x)$. Clearly, we have $0 \leq a \leq A \leq B \leq b$ and $B - A \geq (1 - \delta')(b - a)$. Since $(1 + 2\delta'(1 - \delta')) \geq 0$, we have that $(1 + 2\delta')(B - A) \geq b - a$.

By Lemma 6.2 below, we have that $\sqrt{A} - \sqrt{a} \leq (\varepsilon/2)U$, and $\sqrt{b} - \sqrt{B} \leq (\varepsilon/2)U$, where $U = \sqrt{B} - \sqrt{A}$. Namely, $\sqrt{B} - \sqrt{A} \geq (1 - \varepsilon)(\sqrt{b} - \sqrt{a})$. Namely, $\mathcal{G}$ is a $\varepsilon$-coreset for the extent of $\mathcal{F}$.

The bounds on the size of $\mathcal{G}$ and the running time are easily verified.

**Lemma 6.2** Let $0 \leq a \leq A \leq B \leq b$, and $0 < \varepsilon \leq 1$ be given parameters, so that $b - a \leq (1 + \delta)(B - A)$, where $\delta = \varepsilon^2/16$. Then, $\sqrt{A} - \sqrt{a} \leq (\varepsilon/2)U$, and $\sqrt{b} - \sqrt{B} \leq (\varepsilon/2)U$, where $U = \sqrt{B} - \sqrt{A}$.

**Proof:** Clearly,

$$\sqrt{A} + \sqrt{B} \leq \sqrt{a} + \sqrt{A - a} + \sqrt{b} \leq \sqrt{a} + \sqrt{\delta b} + \sqrt{b} \leq (1 + \sqrt{\delta})(\sqrt{a} + \sqrt{b}).$$

Namely, $\frac{\sqrt{A} + \sqrt{B}}{1 + \sqrt{\delta}} \leq \sqrt{a} + \sqrt{b}$. On the other hand,

$$\sqrt{b} - \sqrt{a} = \frac{b - a}{\sqrt{b} + \sqrt{a}} \leq \frac{(1 + \delta)(B - A)}{\sqrt{b} + \sqrt{a}} \leq (1 + \delta)(1 + \sqrt{\delta})(\sqrt{b} - \sqrt{a}) \leq (1 + \varepsilon/2)(\sqrt{b} - \sqrt{a}).$$
7 Applications

7.1 Minimum Width Annulus

Let $P = \{p_1, \ldots, p_n\}$ be a set of $n$ points in the plane. Let $f_i(q)$ denote the distance of the $i$th point from the point $q$. It is easy to verify that $f_i(q) = \sqrt{(x_q - x_{p_i})^2 + (y_q - y_{p_i})^2}$. Let $F = \{f_1, \ldots, f_n\}$. It is easy to verify that for a center point $x \in \mathbb{R}^2$, the width of the minimum width annulus containing $P$ which is centered at $x$ has width $\mathcal{E}_F(x)$. Thus, we would like to compute a $\varepsilon$-coreset for $F$.

Consider the set of functions $F^2$. Clearly, $f_i^2(x, y) = (x - x_{p_i})^2 + (y - y_{p_i})^2 = x^2 - 2x_{p_i}x + x_{p_i}^2 + y^2 - 2y_{p_i}y + y_{p_i}^2$. Clearly, all the functions of $F^2$ have this (additive) common factor of $x^2 + y^2$. Since we only care about the vertical extent, we have $\mathcal{H} = \{-2x_{p_i}x + x_{p_i}^2 - 2y_{p_i}y + y_{p_i}^2 \mid i = 1, \ldots, n\}$ has the same extent as $F^2$; formally, for any $x \in \mathbb{R}^2$, we have $\mathcal{E}_{F^2}(x) = |\mathcal{E}_F(x)|$. Now, $\mathcal{H}$ is just a family of hyperplanes in $\mathbb{R}^3$, and it has a $\varepsilon^2/64$-coreset $S_{\mathcal{H}}$ for the extent of size $1/\varepsilon$ which can be computed in $O(n + 1/\varepsilon^3)$ time. This corresponds to a $\varepsilon^2/64$-coreset $S_{F^2}$ of $F^2$. By Theorem 6.1, this corresponds to a $\varepsilon$-coreset $S_F$ of $F$. Finally, this corresponds to coreset $S \subseteq P$ of size $O(1/\varepsilon)$, such that the minimum width annulus of $S$, if we expand it by $(1 + 2\varepsilon)$, it contains all the points of $P$. Thus, we can just find the minimum width annulus of $S$. This can be done in $O(1/\varepsilon^2)$ time using an exact algorithm. Putting everything together, we get:

**Theorem 7.1** Let $P$ be a set of $n$ points in the plane, and let $0 \leq \varepsilon \leq 1$ be a parameter. One can compute a $(1 + \varepsilon)$-approximate minimum width annulus to $P$ in $O(n + 1/\varepsilon^3)$ time.

8 From previous lectures

**Theorem 8.1** Let $P$ be a set of $n$ points in $\mathbb{R}^d$, and let $0 < \varepsilon < 1$ be a parameter. One can compute a $\varepsilon$-coreset $S$ for directional width of $P$. The size of the coreset is $O(1/\varepsilon^{d-1})$, and construction time is $O(n + \min(n, 1/\varepsilon^{d-1}))$.

**Theorem 8.2** Let $P$ be a set of $n$ points in $\mathbb{R}^d$. One can compute a $\varepsilon$-coreset for directional width of $P$ in $O(n + 1/\varepsilon^{3(d-1)/2})$ time. The coreset size is $O(\varepsilon^{(d-1)/2})$.

9 Exercises

Exercise 9.1 Prove Lemma 3.1

10 Bibliographical notes

Linearization was widely used in fields such as machine learning [CS00] and computational geometry [AM94].

There is a general technique for finding the best possible linearization (i.e., a mapping $\eta$ with the target dimension as small as possible), see [AM94] for details.

References