Low Dimensional Linear Programming

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1 A game of minimum

Let $x_1, \ldots, x_n$ be $n$ real numbers, and let $y_1, \ldots, y_n$ be a random permutation of them. Let 

$$ \alpha_i = \min_{k=1}^i y_k, $$

and let $M$ be the number of times $\alpha_i \neq a_{i+1}$ for $i = 1, \ldots, n$.

Q: What is the expected value of $M$?
Q: What is the probability that $\Pr \alpha_i \neq \alpha_{i+1} = \frac{1}{i+1}$ and as such $M = \sum_i \frac{1}{i+1} = O(\log n)$,

2 Linear programming in constant dimension ($d > 2$)

Let assume that we have a set $H$ of $n$ linear inequalities defined over $d$ ($d$ is a small constant) variables. Every inequality in $H$ defines a closed half space in $\mathbb{R}^d$. Given a vector $\vec{c} = (c_1, \ldots, c_d)$ we want to find $p = (p_1, \ldots, p_d) \in \mathbb{R}^d$ which is in all the half spaces $h \in H$ and $f(p) = \sum_i c_i p_i$ is maximized. Formally:

<table>
<thead>
<tr>
<th>LP in $d$ dimensions: $(H, \vec{c})$</th>
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</thead>
<tbody>
<tr>
<td>$H$ - set of $n$ closed half spaces in $\mathbb{R}^d$</td>
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<tr>
<td>$\vec{c}$ - vector in $d$ dimensions</td>
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<tr>
<td>Find $p \in \mathbb{R}^d$ s.t. $\forall h \in H$ we have $p \in h$ and $f(p)$ is maximized.</td>
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<tr>
<td>Where $f(p) = \langle p, \vec{c} \rangle$.</td>
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A closed half space in $d$ dimensions is defined by the inequality:

$$ a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \leq b_n. $$

One difficulty that we ignored earlier, is that the optimal solution for the LP might be unbounded:

Namely, we can find a solution with value $\infty$ to the target function.
For a half space $h$ let $\eta(h)$ denote the normal of $h$ directed into the feasible region. Let $\mu(h)$ denote the closed half space, resulting from $h$ by translating it so that it passes through the origin. Let $\mu(H)$ be the resulting set of half spaces from $H$.

The new set of constraints $\mu(H)$ look like:

Lemma 2.1 $(H, \overrightarrow{c})$ is unbounded if and only if $(\mu(H), \overrightarrow{c})$ is unbounded.

Proof: Consider the $\rho'$ the unbounded ray in the feasible region of $(H, \overrightarrow{c})$ such that the line that contain it passes through the origin. Clearly, $\rho'$ is unbounded also in $(H, \overrightarrow{c})$, and this is if and only if.

Lemma 2.2 Deciding if $(\mu(H), \overrightarrow{c})$ is bounded can be done by solving a $d-1$ dimensional LP. Furthermore, if it is bounded, then we have a set of $d$ constraints, such that their intersection prove this.

Furthermore, the corresponding set of $d$ constraints in $H$ testify that $(H, \overrightarrow{c})$ is bounded.

Proof: Rotate space, such that $\overrightarrow{c}$ is the vector $(0, 0, \ldots, 0, 1)$. And consider the hyperplane $g \equiv x_d = 1$. Clearly, $(\mu(H), \overrightarrow{c})$ is unbounded if and only if the region $g \cap \bigcap_{h \in \mu(H)} h$ is non-empty. By deciding if this region is unbounded, is equivalent to solving the following
LP: $L' = (H', (1, 0, \ldots, 0))$ where

$$H' = \left\{ g \cap h \mid h \in \mu(H) \right\}.$$

Let $h \equiv a_1x_1 + \cdots + a_dx_d \leq 0$, the region corresponding to $g \cap h$ is $a_1x_1 + \cdots + a_{d-1}x_{d-1} \leq -a_d$ which is a $d-1$ dimensional hyperplane.

But this is a $d-1$ dimensional LP, because everything happens on the hyperplane $x_d = 1$.

Notice that if $(\mu(H), \overrightarrow{c})$ is bounded (which happens if and only if $(H, \overrightarrow{c})$ is bounded), then $L'$ is infeasible, and the LP $L'$ would return us a set $d$ constraints that their intersection is empty. Interpreting those constraints in the original LP, results in a set of constraints that their intersection is bounded in the direction of $\overrightarrow{c}$.

(In the above example, $\mu(H) \cap g$ is infeasible because the intersection of $\mu(h_2) \cap g$ and $\mu(h_1) \cap g$ is empty, which implies that $h_1 \cap h_2$ is bounded in the direction $\overrightarrow{c}$ which we care about. The positive $y$ direction in this figure.)

We are now ready to show the algorithm for the LP for $L = (H, \overrightarrow{c})$. By solving a $d-1$ dimensional LP we decide whether $L$ is unbounded. If it is unbounded, we are done (we also found the unbounded solution, if you go carefully through the details).

(in the above figure, we computed $p$.)

In fact, we just computed a set $h_1, \ldots, h_d$ s.t. their intersection is bounded in the direction of $\overrightarrow{c}$ (that's what the boundness check returned).

Let us randomly permute the remaining half spaces of $H$, and let $h_1, h_2, \ldots, h_d, h_{d+1}, \ldots, h_n$ be the resulting permutation.
Let $v_i$ be the vertex realizing the optimal solution for the LP:

$$L_i = (\{h_1, \ldots, h_i\}, \vec{c})$$

There are two possibilities:

1. $v_i = v_{i+1}$. This means that $v_i \in h_{i+1}$ and it can be checked in constant time.

2. $v_i \neq v_{i+1}$. It must be that $v_i \notin h_{i+1}$ but then, we must have...

$B$ - the set of $d$ constraints that define $v_{i+1}$. If $h_{i+1} \notin B$ then $v_i = v_{i+1}$. As such, the probability of $v_i \neq v_{i+1}$ is roughly $d/i$ because this is the probability that one of the elements of $B$ is $h_{i+1}$. Indeed, fix the first $i+1$ elements, and observe that there are $d$ elements that are marked (those are the elements of $B$). Thus, we are asking what is the probability of one of $d$ marked elements to be the last one in a random permutation of $h_{d+1}, \ldots, h_{i+1}$, which is exactly $d/(i+1-d)$.

Note that if some of the elements of $B$ is $h_1, \ldots, h_d$ than the above expression just decreases (as there are less marked elements).

Well, let us restrict our attention to $\partial h_{i+1}$. Clearly, the optimal solution to $L_{i+1}$ on $h_{i+1}$ is the required $v_{i+1}$. Namely, we solve the LP $L_{i+1} \cap h_{i+1}$ using recursion.

This takes $T(i+1, d-1)$ time. What is the probability that $v_{i+1} \neq v_i$?

Well, one of the $d$ constraints defining $v_{i+1}$ has to be $h_{i+1}$. The probability for that is $\leq 1$ for $i \leq 2d-1$, and it is

$$\leq \frac{d}{i+1-d},$$

otherwise.

Summarizing everything, we have:

$$T(n, d) = O(n) + T(n, d-1) + \sum_{i=d+1}^{2d} T(i, d-1)$$

$$+ \sum_{i=2d+1}^{n} \frac{d}{i+1-d} T(i, d-1)$$

What is the solution of this monster? Well, one essentially to guess the solution and verify it. To guess solution, let us “simplify” (incorrectly) the recursion to:
\[ T(n, d) = O(n) + T(n, d - 1) + d \sum_{i=2d+1}^{n} \frac{T(i, d - 1)}{i + 1 - d} \]

So think about the recursion tree. Now, every element in the sum is going to contribute a near constant factor, because we divide it by (roughly) \( i + 1 - d \) and also, we are guessing the optimal solution is linear/near linear.

In every level of the recursion we are going to penalized by a multiplicative factor of \( d \). Thus, it is natural, to conjecture that \( T(n, d) \leq (3d)^3 n \).

Which can be verified by tedious substitution into the recurrence, and is left as exercise.

**Theorem 2.3** Given an \( d \) dimensional LP \((H, \overrightarrow{c})\), it can be solved in expected \( O((3d)^3 n) \) time (the constant in the \( O \) is dim independent).

BTW, we are being a bit conservative about the constant. In fact, one can prove that the running time is \( d! n \). Which is still exponential in \( d \).

```plaintext
SolveLP((H, \overrightarrow{c}))
/* initialization */
   Rotate (H, \overrightarrow{c}) s.t. \overrightarrow{c} = (0, \ldots, 1)
   Solve recursively the \( d - 1 \) dim LP:
   \( L' \equiv \mu(H) \cap (x_d = 1) \)
   if \( L' \) has a solution then
      return “Unbounded”
   /* the algorithm itself */
   for \( i \leftarrow d + 1 \) to \( n \) do
      if \( v_{i-1} \in h_i \) then
         \( v_i \leftarrow v_{i-1} \)
      else
         \( v_i \leftarrow \text{SolveLP}((H_{i-1} \cap \partial h_i, \overrightarrow{c})) \) (*)
         where \( H_{i-1} = \{h_1, \ldots, h_{i-1}\} \)
   return \( v_n \)
```

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3 Handling Infeasible Linear Programs

In the above discussion, we glossed over the question of how to handle LPs which are infeasible. This requires slightly modifying our algorithm to handle this case, and I am only describing the required modifications.

First, the simplest case, where we are given an LP $L$ which is one dimensional (i.e., defined over one variable). Clearly, we can solve this LP in linear time (verify!), and furthermore, if there is no solution, we can return two input inequality $ax \leq b$ and $cx \geq d$ for which there is no solution together (i.e., those two inequalities [i.e., constraints] testifies that the LP is not satisfiable).

Next, assume that the algorithm $\text{SolveLP}$ when called on a $d - 1$ dimensional LP $L'$, if $L'$ is not feasible it return the $d$ constraints of $L'$ that together have non-empty intersection. Namely, those constraints are the witnesses that $L'$ is infeasible.

So the only place, where we can get such answer, is when computing $v_i$ (in the (*) line in the algorithm). Let $h'_1, \ldots, h'_d$ be the corresponding set of $d$ constraints of $H_{i-1}$ that testifies that ($H_{i-1} \cap \partial h_i, \overrightarrow{c}$) is an infeasible LP. Clearly, $h'_1, \ldots, h'_d, h_i$ must be a set of $d + 1$ constraints that are together are infeasible, and thats what $\text{SolveLP}$ returns.

4 References

The description in this class notes is loosely based on the description of low dimensional LP in the book of de Berg et al. [dBvKOS00].

References