1 Turán's theorem

1.1 Statement & proof

I think the following proof is due to Alon and Spencer.

Theorem 1.1 (Turán's theorem) *Let $G = (V, E)$ be a graph. The graph G has an independent set of size $\dfrac{n}{1 + d_G}$, where $n = |V|$ and d_G is the average vertex degree in G.*

Proof: Let $\pi = (\pi_1, \ldots, \pi_n)$ be a random permutation of the vertices of G. Pick the vertex π_i into the independent set if none of its neighbors appear before it in π. Clearly, v appears in the independent set if and only if it appears in the permutation before all its $d(v)$ neighbors. The probability for this is $1/(1 + d(v))$. Thus, the expected size of the independent set is (exactly)

$$\tau = \sum_{v \in V} \frac{1}{1 + d(v)}, \tag{1}$$

by linearity of expectations. Thus, by the probabilistic method, there exists an independent set in G of size at least τ.

We remain with the task of proving that $\tau \geq \frac{n}{1+d_G}$. Observe that if $x + y = \alpha$, then

$$\frac{1}{1+x} + \frac{1}{1+y} = \frac{1+x+1+y}{1+x+y+xy} = \frac{2+\alpha}{1+\alpha+xy} \geq \frac{2+\alpha}{1+\alpha+\alpha^2/4} = \frac{2(1+\alpha/2)}{(1+\alpha/2)^2} = \frac{2}{1+\alpha/2},$$

since the quantity xy is maximized when $x = y$ under the condition $x + y = \alpha$. This implies that the minimum of Eq. (1) is achieved if we replace $d(v)$ by the average degree in G, which implies the theorem. ■

Following a post of this write-up on my blog, readers suggested two modifications. We present an alternative proof incorporating both suggestion.

Alternative proof of Theorem 1.1: We associate a charge of size $1/(d(v) + 1)$ with each vertex of G. Let $\gamma(G)$ denote the total charge of the vertices of G. We prove, using induction, that there is always an independent set in G of size at least $\gamma(G)$. If G is the empty graph, then the claim trivially holds. Otherwise, assume that it holds if the graph has at most $n - 1$ vertices, and consider the vertex v of lowest degree in G. The total charge of v and its neighbors is

$$\frac{1}{d(v)+1} + \sum_{uv \in E} \frac{1}{d(u)+1} \leq \frac{1}{d(v)+1} + \sum_{uv \in E} \frac{1}{d(v)+1} = \frac{d(v)+1}{d(v)+1} = 1,$$

since $d(u) \geq d(v)$, for all $uv \in E$. Now, consider the graph H resulting from removing v and its neighbors from G. Clearly, $\gamma(H)$ is larger (or equal) to the total charge of the vertices of $V(H)$ in G, as their degree had either decreased (or remained the same). As such, by induction, we have an independent set in H of size at least $\gamma(H)$. Together with v this forms an independent set in G of size at least $\gamma(H) + 1 \geq \gamma(G)$. Implying that there exists an independent set in G of size

$$\tau = \sum_{v \in V} \frac{1}{1 + d(v)}, \tag{2}$$

Now, set $x_v = 1 + d(v)$, and observe that

$$(n + 2|E|)\,\tau = \left(\sum_{v\in V} x_v\right)\left(\sum_{v\in V} \frac{1}{x_v}\right) \geq \sum_{v\in V} x_v \frac{1}{x_v} = n.$$

Namely, $\tau \geq \dfrac{n}{n+2|E|} = \dfrac{1}{1+2|E|/n} = \dfrac{1}{1+d_G}$. ■

1.2 An algorithm for the weighted case

In the weighted case, we associate weight $w(v)$ with each vertex of G, and we are interested in the maximum weight independent set in G. Deploying the algorithm described in the first proof of Theorem 1.1, implies the following.

Lemma 1.2 *The graph $G = (V, E)$ has an independent set of size* $\geq \sum_{v\in V} \dfrac{w(v)}{1+d(v)}$.

Proof: By linearity of expectations, we have that the expected weight of the independent set computed is equal to

$$\sum_{v\in V} w(v) \cdot \mathbf{Pr}[v \text{ in the independent set}] = \sum_{v\in V} \frac{w(v)}{1+d(v)},$$

■