Fast Construction of Nets in Low Dimensional Metrics, and Their Applications

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k-Median

- Given: Graph $G = (V, E)$, $k \in \mathbb{N}$.
- Finite metric: $d_G()$ - shortest distance in graph.
- Find: Centers $c_1, \ldots, c_k \in V$, minimizing $\sum_{v \in V} \min_{i \in \{1,\ldots,k\}} d_G(v, c_i)$.
- NP-Hard optimization problem.
- $\exists$ Poly-time $3 + \varepsilon$ approx algorithm **[Arya et al., 2001]**.
- Can do much better in low-dim Euclidean settings...
- Q: $\exists$ cases such that finite metric $\approx$ Euclidean metric?

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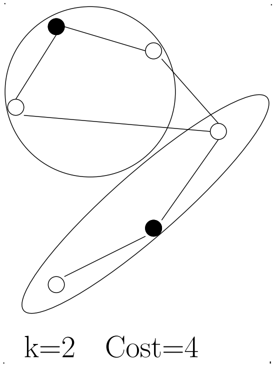
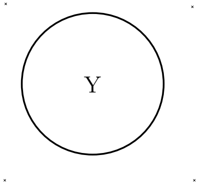


Doubling Dimension

Definition (Doubling Dimension ddim(X))

The least ρ, $\forall Y \subset X\ \exists Z_1, \ldots Z_{2^\rho}$, $\mathrm{diam}(Z_i) \leq \frac{\mathrm{diam}(Y)}{2}$ and $\bigcup_i Z_i \supset Y$.



- $\mathrm{ddim}(\mathbb{R}^d) = \Theta(d)$.
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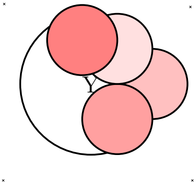

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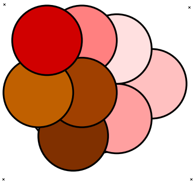

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1. Naturally extends Euclidean settings.
2. Truely extended Euclidean metrics.
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Why work on finite metric spaces?

- Strong (geometric) intuition.
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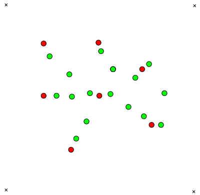
History of Doubling Metric

- Related to Hausdorff-Besicovitch dimension.
- Used in the Geometric Analysis community **[Assouad, 1983]**.
- Introduced to CS by **[Gupta et al., 2003]**. (Also **[Clarkson, 1999]**.)
- Approx. Nearest Neighbor Search (ANNS) **[Krauthgamer and Lee, 2004b, Krauthgamer and Lee, 2004a]**
- Quasi PTAS for TSP **[Talwar, 2004]**.
- Efficient distance labeling **[Talwar, 2004, Slivkins, 2004]**

Nets in Metric Spaces

r-net S:

- Balls of radius r centered at S cover the space.
- Balls of radius $r/2$ centered at S are packed in the space.



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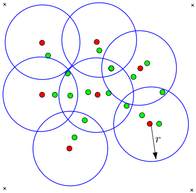

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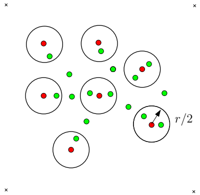

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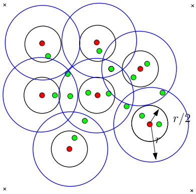

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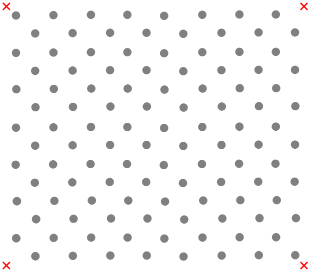

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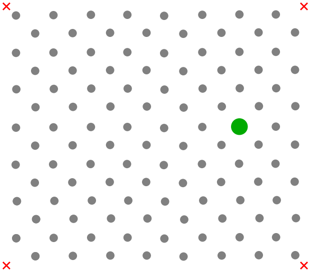

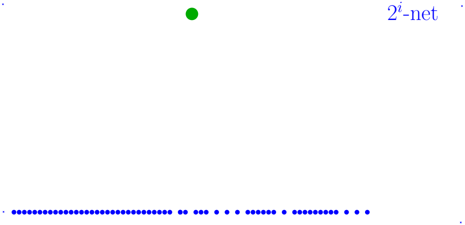

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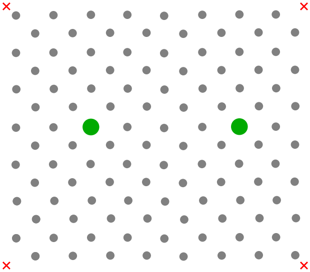

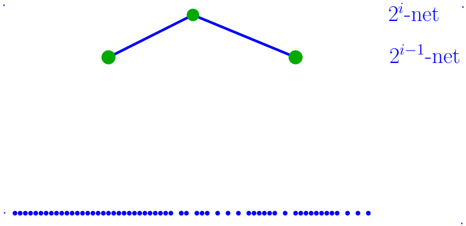

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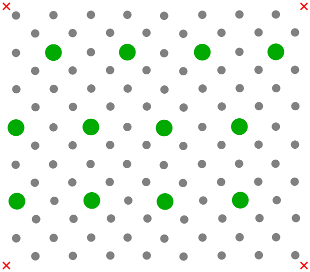

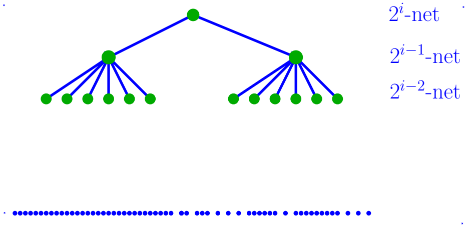

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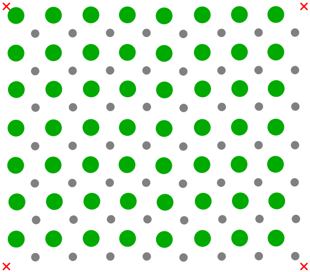

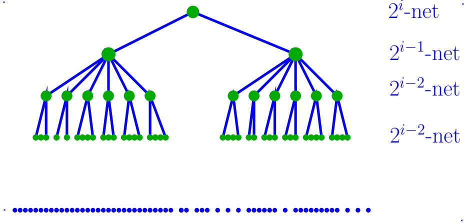

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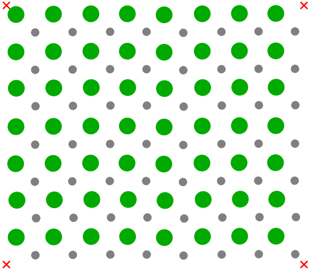

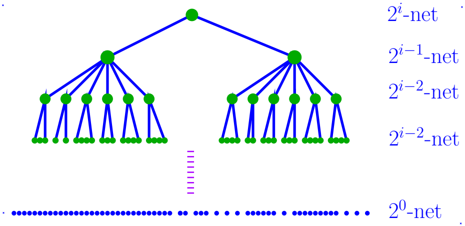

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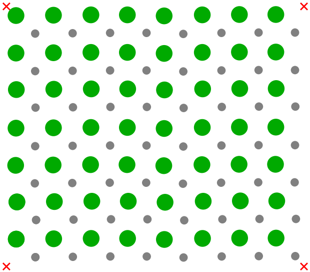

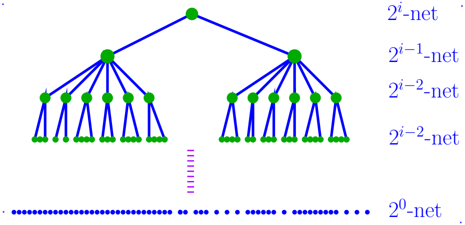

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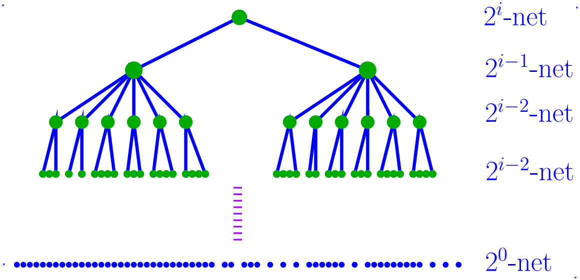
Nets and ANNS



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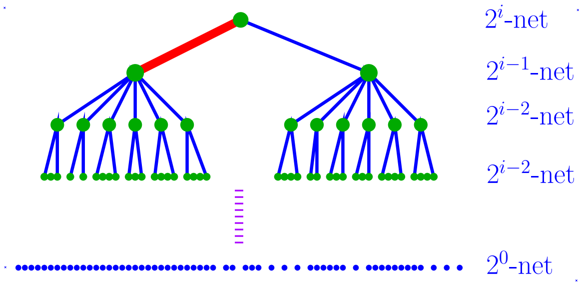

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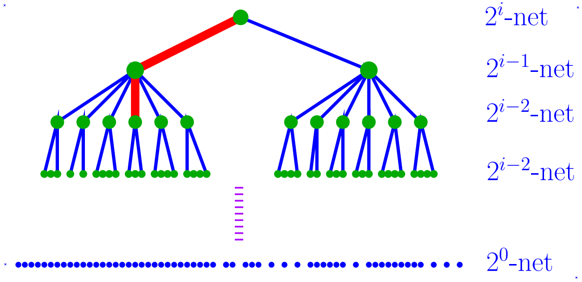

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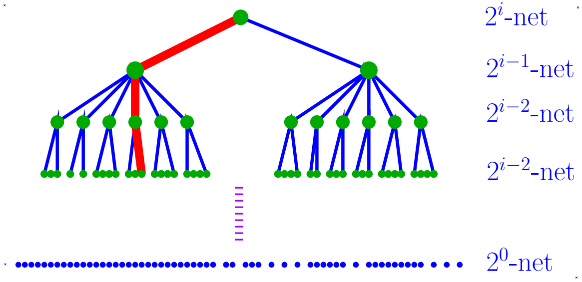

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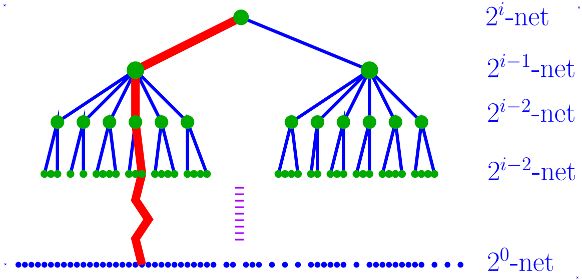

Main Technical Result

Theorem

A hierarchy of nets at all scales can be constructed in $2^{O(ddim)} n \log n$ *time.*

Remark

The input size is $\Theta(n^2)$. The alg' is randomized and uses sampling.

Observation

Ideas leads to simpler construction of compressed quadtrees.

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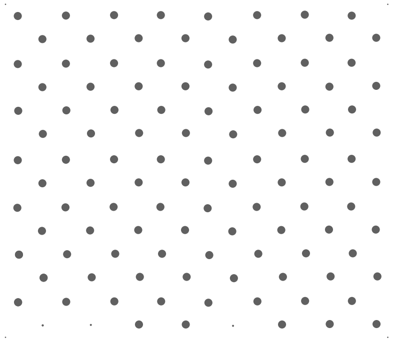

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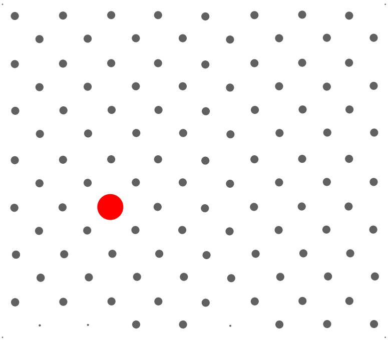

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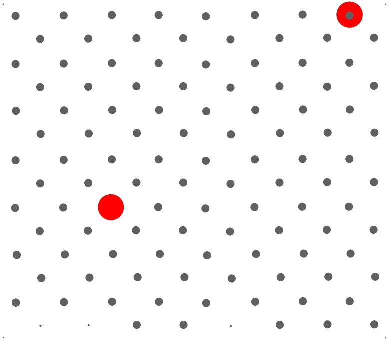

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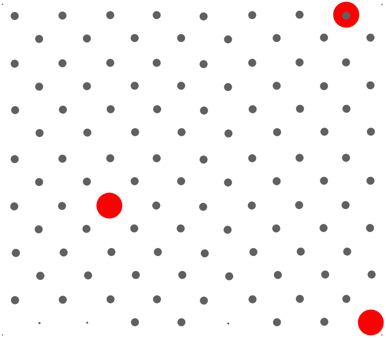

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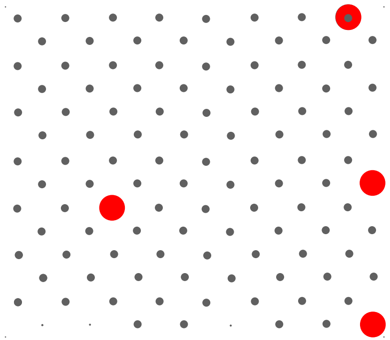

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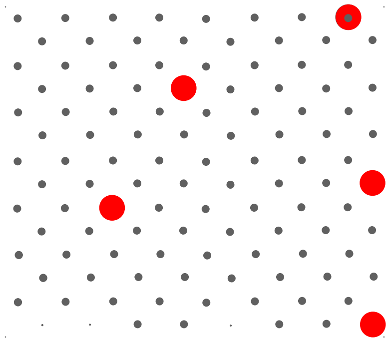

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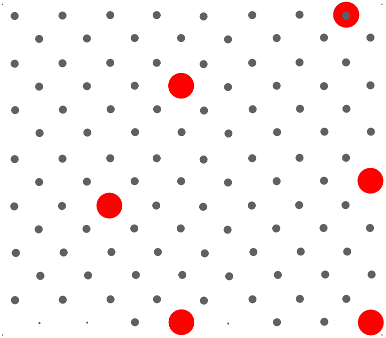

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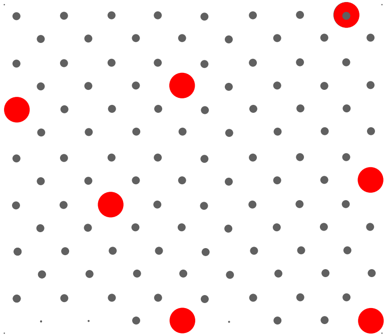

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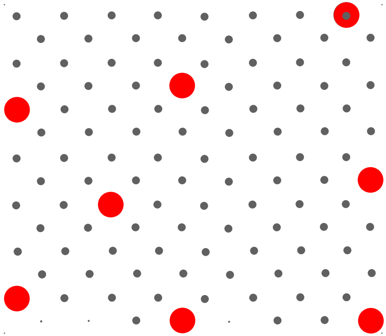

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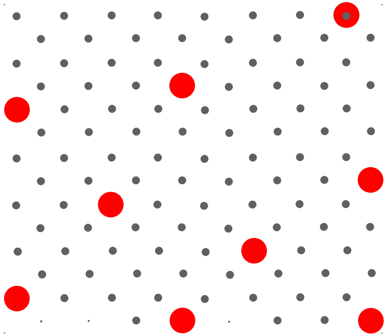

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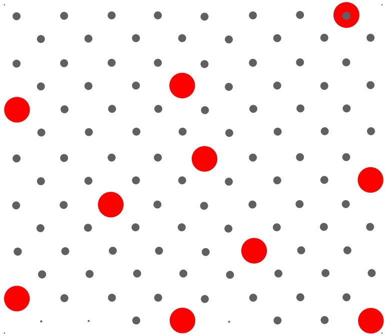

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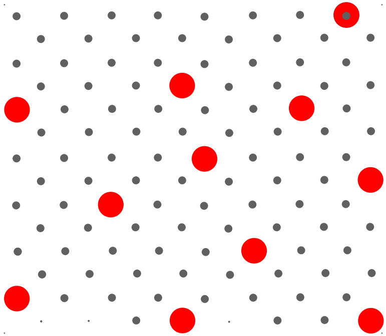

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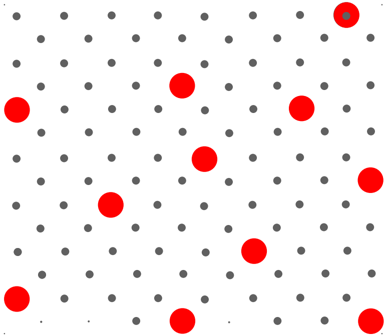

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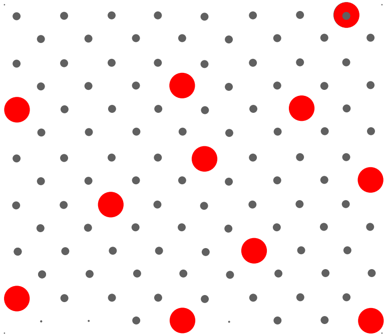

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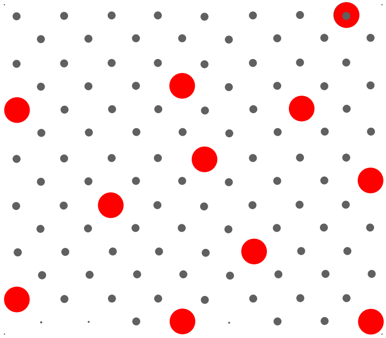

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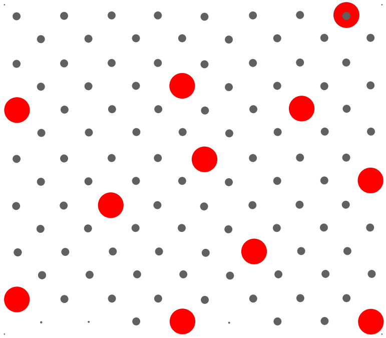

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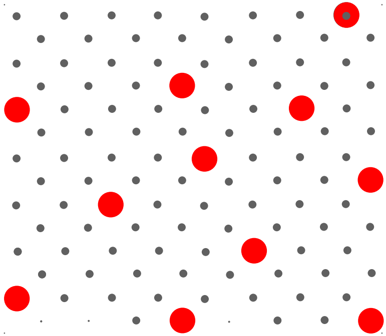

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3. "invented" by **[Gonzalez, 1985]** takes $O(n^2)$ time.
4. New result: compute sequence in $O(n \log n)$ time.
5. Not trivial... Nasty issues with spread.

Aprox. Nearest Neighbor in Doubling Metrics

Theorem

Any n-point metric with fixed doubling dimension admits ANNS with
Approx $= 1 + \varepsilon$, *Query* $= O(\varepsilon^{-\mathrm{ddim}} + \log n)$,
Space $= O(n)$, *Preprocessing time* $= O(n \log n)$.

- Basic idea appears already in [Clarkson, 1999] (without the analysis and fast construction).
- Similar to results known for $\mathbb{R}^d$.
- Improves over:
  - [Krauthgamer and Lee, 2004b] - dependency on spread.
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Well-separated pairs decomposition

- P: set of n points with ddim dim.
- Result: Compute WSPD in near linear time.
- # of pairs is $n\varepsilon^{-O(\mathsf{dim})}$
- Improves **[Talwar, 2004]** work (spread).
- Matches the results of **[Callahan and Kosaraju, 1995]**

Spanners.

- $1 + \varepsilon$-spanners
- number of edges $n\varepsilon^{-O(\mathrm{dim})}$
- Similar result by **[Chan et al., 2005]**.

Compact Representation Scheme (CRS).

- Build a data-structure.
- linear size
- approx. distance queries between pairs of points in constant time.
- “approximate distance oracles” **[Thorup and Zwick, 2001]**
- Extends **[Gudmundsson et al., 2002a, Gudmundsson et al., 2002b]** (geometric settings)
- Improve/unify: **[Talwar, 2004, Slivkins, 2005]**.

A variant of Assouad's embedding.

Theorem

$\mathcal{M}$: *metric space with doubling dim* $\dim$
Can be embedded in ℓ_∞^d.
Where $d \leq \varepsilon^{-O(\dim)}$
$(\mathcal{M}, \sqrt{d_\mathcal{M}})$ *is distorted by* $1+\varepsilon$ *factor.*

Embedding can be computed quickly...
[Indyk and Naor, 2005]
Better dimension reduction for such cases.

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Doubling Measure.

- A doubling measure μ:
  $x \in P$ and $r > 0$, the ratio $\mu(\mathbf{b}(x, 2r))/\mu(\mathbf{b}(x, r))$ is bounded.
  $\mathbf{b}(x, r) = \{y : \; d(x, y) \le r\}$.
- Another definition of dimension. Weaker.
- **[Vol′berg and Konyagin, 1987]**
  Any doubling-dim metric can be converted to be doubling measure.
  Weights of points exponential.
- Present a near linear time algorithm for computing it.

Lipschitz Constant of a Mapping.

- $f : P \to B$
- Compute Lipschitz constant of f?
- Approx follows from WSPD.
  Near linear time.
- $P \subseteq \mathbb{R}^2$ or $P \subseteq \mathbb{R}$:
  Compute Lipschitz constant exactly in near linear time.

Computing the Doubling Dimension.

- $\mathcal{M}$: Finite metric space.
- Q: What is the doubling dim of $\mathcal{M}$?

New result

One can compute $O(\dim(\mathcal{M}))$ in $2^{O(\dim)} n \log n$ time

No need to know the doubling dim in advance...

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Open problems

- Does doubling metric really exists in nature?
  Temporary answer: yes and no.
- Simplify net-tree construction?
- Good spanners - light and low diam?
- What is dimension?
  (Recent survey by Clarkson.)

Thank You

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