Approximation Algorithms for Maximum Independent Set of Pseudo-Disks

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2: Independent Set Problem

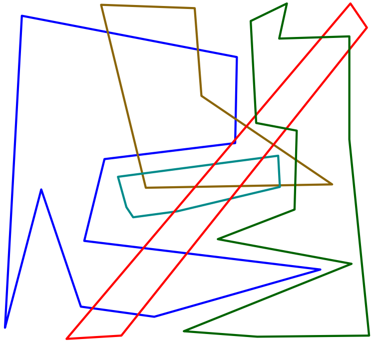
$\mathbf{F} = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$: $\mathbf{n}$ objects in $\mathbb{R}^2$.
$\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n > \mathbf{0}$: weights.

Problem (Independent set.)

Find a maximum weight subset of $\mathbf{F}$ *s.t. no pair intersects.*

Fundamental problem...

1. Packing.
2. Frequency assignment/coloring.
3. ...



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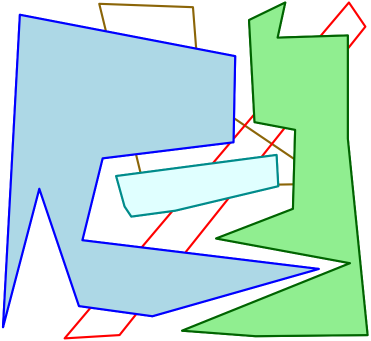


3: Independent Set Problem

Background

- $\mathbf{G = (V, E)}$: intersection graph
  Find maximum independent set in $\mathbf{G}$
- Indep set is **NP-Complete**.
- **[Hastad, 1996]**
  No approximation $|\mathbf{V}|^{1-\varepsilon}$ if NP $\neq$ ZPP .
- **[Berman and Fujito, 1999]**
  If max. degree of the graph is $\mathbf{3} \implies$ no PTAS

4: Independent Set Problem

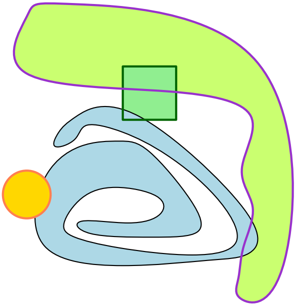
Background - geometric settings

1. **[Chan, 2003, Erlebach et al., 2005]**
   PTAS for independent set of fat objects (weighted).
   Quadtree + dynamic programming techniques
2. **[Chan, 2003]**
   Using separator theorem - unweighted.
3. **[Chalermsook and Chuzhoy, 2009]**
   axis-parallel rectangles – $O(\log \log n)$-approximation
4. **[Agarwal and Mustafa, 2006]**
   segments – roughly $O(\sqrt{\mathrm{Opt}})$-approximation

5: Pseudo-disks

Definition

A set of objects is a collection of **pseudo-disks**, if the boundary of every pair of them intersects at most twice.



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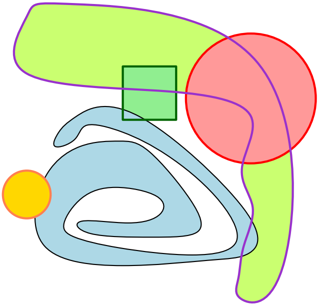

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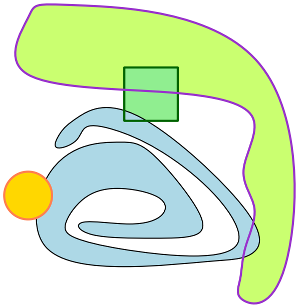
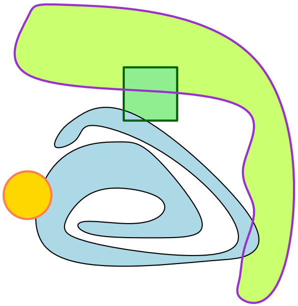


Nature channel summary.

1. Herd animal.
2. Roam the plane.

6: Independent Set for Pseudo-disk

- Previous techniques seem powerless! Feeble even!
- **Greedy algorithm**: Pick smallest object to independent set, and repeat.
  - Constant factor for: disks/fat objects.
  - Pseudo-disks: FAIL!



7: Local Search for Independent Set

- Much work on local search
  Survey: **[Halldórsson, 1998]**.
- $\boldsymbol{\Delta}$: max degree of a graph
  $\implies \boldsymbol{\Delta}/\mathbf{4}$ approximation via local search.
- **[Agarwal and Mustafa, 2006]**
  constant approximation for pseudo-disks that are rectangles.

New result

PTAS for unweighted pseudo-disks.

When applicable: slower but simpler than prev algorithms.

Essentially same algorithm/analysis as...

PTAS for geometric hitting set problem via local search
Nabil H. Mustafa and Saurabh Ray
SoCG 2009.

Their result:

1. "dual" to ours.
2. harder...
3. independently independent.

Theorem

For a weighted set of pseudo-disks, computes $\mathbf{O(1)}$*-approximate independent set.*

Thesis motivating this work

LP is a powerful tool that can help solving problems in low dim CG.

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10: A comment on methodology

Time class
Doubly exponential
Exponential / **NP-Hard**
Polynomial
Quadratic
Near linear
Linear
Sub-linear
Constant
Sub-constant time

(Polynomial through Constant:) "traditional CG"

10: A comment on methodology

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"Operation research" (Doubly exponential, Exponential / NP-Hard, Polynomial)

As such...

- Solving LP is polytime.
- Polynomial time $=$ good.
- Use LP as an "oracle".

11: LP for independent set

A clueless oracle in action.

$\mathbf{S} = \{s_1, \ldots s_n\}$ - segments in the plane.
$x_i \in \{0, 1\}$: var deciding if to include s_i

$$\alpha = \max \sum_i w_i x_i$$
$$0 \leq x_i \leq 1 \quad \forall i$$
$$x_i + x_j \leq 1 \quad \forall i, j \quad \text{if } s_i \cap s_j \neq \emptyset$$

Bad news...

$x_i = 1/2$ for all i. Valid solution: $\alpha = n/2$.
Integrality gap $= n/2$.

12: A rounding scheme.

LP for independent set - another step in the wrong direction.

Given a solution to:

$$\alpha = \max \sum_i w_i x_i$$

$$0 \le x_i \le 1 \quad \forall i$$

$$x_i + x_j \le 1$$

$$\text{if } s_i \cap s_j \neq \emptyset$$

- Put s_i into $\mathcal{R}$ with prob' x_i.
- $\mathbf{E}[\text{ cost of } |\mathcal{R}|] = \sum_i x_i w_i = \text{LP opt}$.
- $\beta = \mathbf{E}[\#\text{ bad pairs } \mathcal{R}] = \sum_{\substack{i<j \\ s_i \cap s_j \neq \emptyset}} x_i x_j$.
- $\mathcal{R}$ induces an intersection graph: $G = (\mathcal{R}, E)$.
- $\mathbf{E}[|E|] = \beta$.
- β small there is hope...

randIndepSet$(G = (V, E))$

```
V = ⟨v_1, ..., v_n⟩ - random
I ← ∅
for i = 1 ... n do
    if Neigberhood(v_i) ∩ I = ∅
        I ← I ∪ {v_i}
return I.
```

We have...

$v \in V$,
$\Pr[v \in I] = \frac{1}{d(v)+1}$.

$$\mathbf{E}\left[\text{size of indep set}\right] = \sum_{v \in V} \frac{1}{d(v)+1}$$

randIndepSet$(G = (V, E))$

$V = \langle v_1, \ldots, v_n \rangle$ - random
$I \leftarrow \emptyset$
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13: Turán's theorem

A "Greedy" algorithm for independent set.

randIndepSet$(G = (V, E))$

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Weighted case: $\mathbf{E}[\text{cost indep set}] = \sum_{v} \frac{w(v)}{d(v)+1}$.

14: Turán's theorem implies...

A good approximation if the graph is sparse.

Theorem (Turán's Theorem)

$\mathbf{G} = (\mathbf{V}, \mathbf{E})$ - *graph.*

$$\mathbf{E}[\textit{cost indep set}] = \sum_{\mathbf{v}} \frac{\mathbf{w(v)}}{\mathbf{d(v)} + 1}.$$

If $\mathbf{d(v)} = \mathbf{O(1)}$ then constant factor approximation.

Intuitively...

If $|\mathbf{E(G)}| = \mathbf{O}(|\mathbf{V(G)}|)$ then we get const approx.

$\mathbf{s_1, \ldots s_n}$: segments.

Given a solution to:

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$$0 \le x_i \le 1 \quad \forall i$$
$$x_i + x_j \le 1$$
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Change the LP...

- Put s_i into $\mathcal{R}$ with prob' x_i.
- $\mathcal{R}$ induced an intersection graph: $G = (\mathcal{R}, E)$.
- $\mathcal{E} = E[|\mathcal{R}|] = \sum_i x_i$: Energy.
- $\beta = E[|E|] = \sum_{i<j, s_i \cap s_j \neq \emptyset} x_i x_j$.
- β might be very large.

15: Back to the rounding scheme

Extracting a sparse intersection graph

$s_1, \ldots s_n$: segments.

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A LP for pseudo-disks

No point is too deep

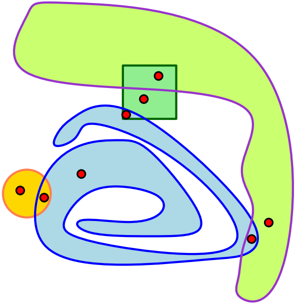
$\mathbf{F} = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$: pseudo-disks in the plane.

New LP

$$\max \sum_{i=1}^{n} w_i x_i$$

$$\sum_{p \in f_i} x_i \leq 1 \quad \forall p \in \mathrm{Verts}(F)$$

$$0 \leq x_i \leq 1, \quad \forall i.$$



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The rounding scheme

$F = \{f_1, \ldots, f_n\}$: pseudo-disks in the plane.

New LP

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$$0 \le x_i \le 1, \quad \forall i.$$

Relevant quantities...

$$\mathcal{E} = E[|\mathcal{R}|] = \sum_i x_i$$

$$\beta = E[|E|] = \sum_{\substack{i<j \\ f_i \cap f_j \ne \emptyset}} x_i x_j.$$

Key lemma.

$\beta = O(\mathcal{E})$.

Average degree of $G = (\mathcal{R}, E)$ is $O(\beta/\mathcal{E}) = O(1)$.

Done, by applying Turán's theorem to $G = (\mathcal{R}, E)$.

18: A proof of the key lemma.

Lemma: $$\sum_{i<j, f_i \cap f_j \neq \emptyset} x_i x_j = O\left(\sum_i x_i\right).$$

Proof.

$p \in \mathrm{Verts}(F)$: $p = \partial f_i \cap \partial f_j$.

Use Clarkson's technique like argument.

Lemma: $$\sum_{i<j, f_i \cap f_j \neq \emptyset} x_i x_j = O\left(\sum_i x_i\right).$$

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p is vertex of union of R if (i) f_i and f_j are in R.
(ii) No object covering p is in R.

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$$\Pr\left[p \in \partial \bigcup R\right] = x_i x_j \prod_{\substack{k \neq i,j \\ p \in \mathrm{int}(f_k)}} (1 - x_k)$$

18: A proof of the key lemma.

bla, bla, bla

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$$\Pr\left[p \in \partial \bigcup R\right] = x_i x_j \prod_{\substack{k \neq i,j \\ p \in \mathrm{int}(f_k)}} (1 - x_k) \geq x_i x_j \exp(-2 \sum_{\substack{k \neq i,j \\ p \in \mathrm{int}(f_k)}} x_k)$$

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Lemma: $\displaystyle\sum_{i<j, f_i \cap f_j \neq \emptyset} x_i x_j = O\left(\sum_i x_i\right).$

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Expected complexity of union of $\mathcal{R}$ is $\tau = O(E[|\mathcal{R}|]) = O(\sum_i x_i)$.

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$$\tau = \sum_p \Pr\left[p \in \partial \bigcup R\right] \geq \sum_{i<j, f_i \cap f_j \neq \emptyset} \frac{x_i x_j}{10}.$$

bla, bla, bla, bla, bla, bla, bla, bla, **bla**, bla, **bla!!!**

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We conclude $\sum_{i<j, f_i \cap f_j \neq \emptyset} \frac{x_i x_j}{10} = O(\sum_i x_i)$.

∎

19: Conclusions

This talk is almost over...

Results

(i) Const approx: independent set of weighted pseudo-disks.

(ii) PTAS for unweighted pseudo-disks

The paper contain some other results...

Open problems

(i) Rectangles?

(ii) Hardness?

20: Thank you...

Skål !

Bunden i vejret eller resten i håret!

English: Bottoms up or the rest in your hair.

Original Viking meaning: Bottoms up or the rest (we drink) from your skull.

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