New Constructions of SSPDs and their Applications

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June 13, 2010

2: What is SSPD?

- Southeastern Society of Pediatric Dentistry.
- Society for Stimulus Properties of Drugs.
- Small-Scale Powder Disperser.
- SSPD 2010 : Sensor Signal Processing for Defence.
- SSPD = Semi-Separated Pairs Decomposition.

2: What is SSPD?

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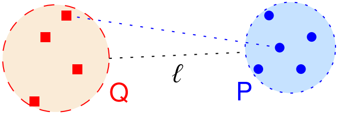
3: Well separated pair



Definition: $1/\varepsilon$-separated pair

$\mathbf{P}, \mathbf{Q} \subseteq \mathbb{R}^{\mathbf{d}}$.

$$\ell \geq \frac{1}{\varepsilon} * \max(\mathrm{diam}(\mathbf{P}) \ , \ \mathrm{diam}(\mathbf{Q}))$$

Captures all the distances $\mathbf{P} \otimes \mathbf{Q}$ up to $1 \pm \varepsilon$.

4: Pairs decomposition

Definition (Pair decomposition.)

For a point-set P, a **pair decomposition** of P is a set of pairs $\mathcal{W} = \left\{ \{\mathcal{X}_1, \mathcal{Y}_1\}, \ldots, \{\mathcal{X}_s, \mathcal{Y}_s\} \right\}$, such that
(i) $\mathcal{X}_i, \mathcal{Y}_i \subset \mathsf{P}$ for every i,
(ii) $\mathcal{X}_i \cap \mathcal{Y}_i = \emptyset$ for every i, and
(iii) $\cup_{i=1}^{s} \mathcal{X}_i \otimes \mathcal{Y}_i = \mathsf{P} \otimes \mathsf{P}$.

The **weight** of a pair decomposition $\mathcal{W}$ is defined to be $\omega(\mathcal{W}) = \sum_{i=1}^{s} (|\mathcal{X}_i| + |\mathcal{Y}_i|)$.

Definition ($1/\varepsilon$-WSPD)

$\mathbf{P}$: set of points.

(A) $\mathcal{W}$: pair decomposition of $\mathbf{P}$.

(B) $\forall \{\mathcal{X}_i, \mathcal{Y}_i\} \in \mathcal{W}$

$\mathcal{X}_i$ and $\mathcal{Y}_i$ are $(1/\varepsilon)$-separated.

Theorem ([Callahan and Kosaraju, 1995])

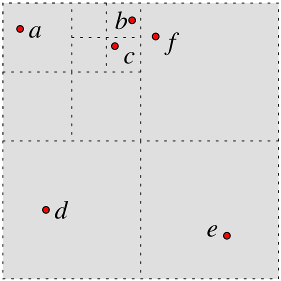
$\mathbf{P} \subseteq \mathbb{R}^d$: set of $\mathbf{n}$ points

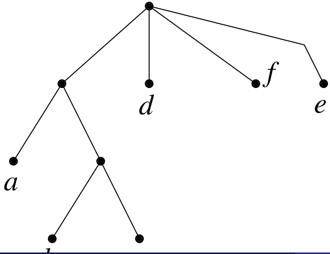
$\exists$ $1/\varepsilon$-WSPD with

(A) $O(n/\varepsilon^d)$ pairs.

(B) computed in $O(n \log n + n/\varepsilon^d)$ time.

The representation



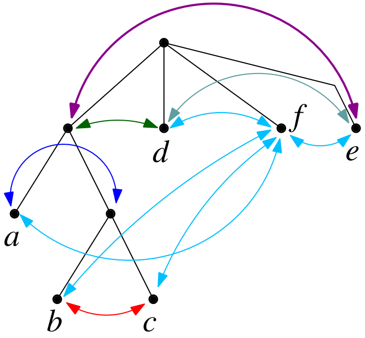


$$A_1 = \{d\},\ B_1 = \{e\}$$
$$A_2 = \{a, b, c\},\ B_2 = \{e\}$$
$$A_3 = \{a, b, c\},\ B_3 = \{d\}$$
$$A_4 = \{a\},\ B_4 = \{b, c\}$$
$$A_5 = \{b\},\ B_5 = \{c\}$$
$$A_6 = \{a\},\ B_6 = \{f\}$$
$$A_7 = \{b\},\ B_7 = \{f\}$$
$$A_8 = \{c\},\ B_8 = \{f\}$$
$$A_9 = \{d\},\ B_9 = \{f\}$$
$$A_{10} = \{e\},\ B_{10} = \{f\}$$

Implicit representation using a tree

$$\mathcal{W} = \left\{ \begin{array}{c} \{\mathbf{A}_1, \mathbf{B}_1\}, \\ \{\mathbf{A}_2, \mathbf{B}_2\}, \\ \{\mathbf{A}_3, \mathbf{B}_3\}, \\ \{\mathbf{A}_4, \mathbf{B}_4\}, \\ \{\mathbf{A}_5, \mathbf{B}_5\}, \\ \{\mathbf{A}_6, \mathbf{B}_6\}, \\ \{\mathbf{A}_7, \mathbf{B}_7\}, \\ \{\mathbf{A}_8, \mathbf{B}_8\}, \\ \{\mathbf{A}_9, \mathbf{B}_9\}, \\ \{\mathbf{A}_{10}, \mathbf{B}_{10}\} \end{array} \right\}$$

$\{A_2, B_2\} \equiv \{a, b, c\} \otimes \{e\}$



8: WSPD- why is the representation implicit?

Definition (Weight.)

$\mathcal{W} = \{ \{\mathcal{X}_1, \mathcal{Y}_1\}, \ldots, \{\mathcal{X}_s, \mathcal{Y}_s\} \}$.
weight: $\omega(\mathcal{W}) = \sum_{i=1}^{s} (|\mathcal{X}_i| + |\mathcal{Y}_i|)$.

Weight $=$ total space to list pair decomposition explicitly.

Observation

$\exists \mathbf{P}$ of $\mathbf{n}$ points s.t. for any WSPD $\mathcal{W}$ of $\mathbf{P}$: $\omega(\mathcal{W}) = \Omega(\mathbf{n}^2)$.

Lemma ([Bollobás and Scott, 2007])

For any pair-decomposition of $\mathbf{P}$: $\omega(\mathcal{W}) = \Omega(\mathbf{n} \log \mathbf{n})$.

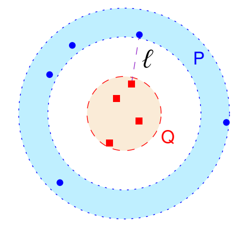
Question?

Is there any "separated" pair-decomposition of small weight?

9: Semi-separated pairs decomposition

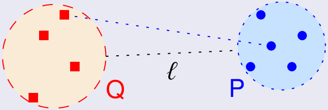
Separated but (not) equal

Definition: $1/\varepsilon$-semi-separated pair



$$\ell \geq \frac{1}{\varepsilon} * \mathbf{min}(\mathrm{diam}(\mathbf{P}) \ , \ \mathrm{diam}(\mathbf{Q}))$$

Definition: $1/\varepsilon$-separated pair



$$\ell \geq \frac{1}{\varepsilon} * \mathbf{max}(\mathrm{diam}(\mathbf{P}) \ , \ \mathrm{diam}(\mathbf{Q}))$$

Definition

$1/\varepsilon$-SSPD $=$ a pair decomposition: all pairs $1/\varepsilon$-semi separated.

- Def. by [Varadarajan, 1998].
  Weight: $O\left(n \log^4 n\right)$
- [Abam, de Berg, Farshi, and Gudmundsson, 2009a]
  2d $=$ weight: $O(\varepsilon^{-2} n \log n)$
- [Abam, de Berg, Farshi, Gudmundsson, and Smid, 2009b]
  Weight: $O(\varepsilon^{-d} n \log n)$
- Constructions uses BAR trees.
  [Duncan, Goodrich, and Kobourov, 2001]

Limitations of previous constructions

- BAR = Bounded aspect ratio
  Uses angles and cutting planes.
- No BAR trees for doubling metrics.
- Point might participate in many pairs.

12: New results

Groundbreaking work of insignificant importance

- **Observation:**

  SSPD is easy for bounded spread case.

  Using quadtrees.

- Ring separation + above
  $\Longrightarrow$ $\mathbf{1}/\varepsilon$-SSPD of weight $\mathbf{O(\varepsilon^{-d} n \log^2 n)}$.
  (Simple!)

- $\mathbf{1}/\varepsilon$-SSPD
  (A) Using random partition (Semi-painful analysis.)
  (B) Weight $\mathbf{O(\varepsilon^{-d} n \log n)}$.
  (C) Each point appears in $\mathbf{O(\varepsilon^{-d} \log n)}$ pairs.
  (D) Number of pairs $\mathbf{O(n/\varepsilon^{d})}$.

13: Applications of New results

Whatever. (That was original.)

- Distance based constructions.
- Work verbatim for doubling metrics.
- Spanners.

Theorem

$\varepsilon > 0$, $\mathbf{P} \subseteq \mathbb{R}^d$, $\mathbf{n} = |\mathbf{P}| \implies (1+\varepsilon)$*-spanner* $\mathbf{G}$*:*

(A) $O(n/\varepsilon^{2d-1})$ *edges,*

(B) Max degree $O\left((1/\varepsilon^{2d-1}) \log^2 n\right)$.

(C) A separator of size $O(n^{1-1/d}/\varepsilon^d)$.

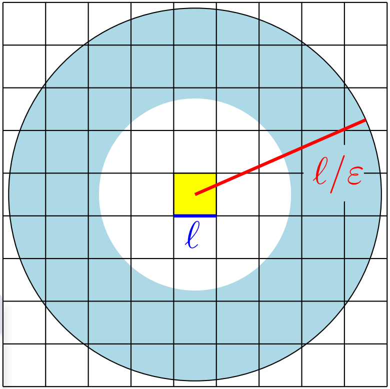
(D) Construction time: $O\left((n/\varepsilon^d) \log^2 n\right)$.

Previous, slightly weaker result:
[Fürer and Kasiviswanathan, 2007].

WSPD for bounded spread

If you build it, they would be separated.

- **P**: Point set with Δ spread.
- Spread = ratio between largest/shortest dist. in $\binom{P}{2}$.
- **T**: Quadtree for **P**.
- Depth of **T**$=$ $O(\log \Delta)$.
- Build WSPD using **T**, all pairs having same level.



Observation

Node of **T** participates in $O(1/\varepsilon^d)$ pairs of WSPD.

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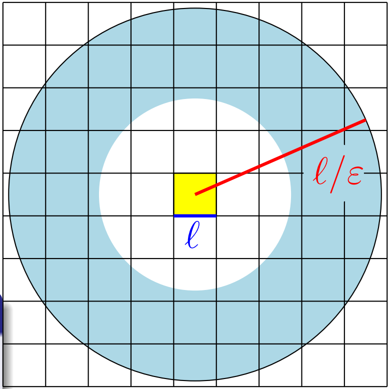


Observation

Node of **T** participates in $\mathbf{O(1/\varepsilon^d)}$ pairs of WSPD.

15: WSPD for bounded spread

If you separate them, they will be upset

Observations

(A) Node of $\mathbf{T}$ participates in $\mathbf{O(1/\varepsilon^d)}$ pairs of WSPD.
(B) Height of quadtree $\mathbf{O(\log \Delta)}$
(C) Total weight of resulting WSPD is $\mathbf{O}\left(\frac{\mathbf{n}}{\varepsilon^{\mathbf{d}}} \log \mathbf{\Delta}\right)$.

Done... if $\mathbf{\Delta = n^{O(1)}}$.

16: Chicken chicken chicken chicken

Chicken chicken chicken chicken chicken chicken chicken chicken chicken chicken chicken chicken chicken chicken chicken chicken chicken

Chicken (**[Chicken, Chicken, 2006]**)

P: **n** chickens in $\mathbb{R}^d$

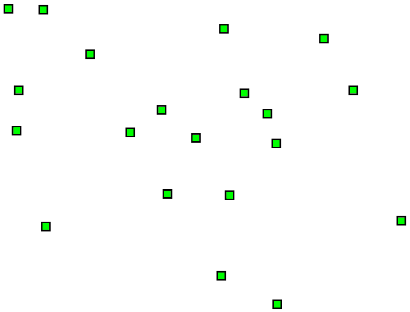
$\Longrightarrow$

(A) $\exists \mathbf{r}, \mathbf{R}$ chicken $\mathbf{r} \leq \mathbf{R}$.

(B) chicken chicken chicken chicken

(C) chicken chicken chicken chicken chick.

(D) chicken chicken!



Chicken $\Longrightarrow$ chicken "chicken".

17: A lemma about point sets in low dimensions

What has been will be again, what has been done will be done again; there is nothing new under the sun.

Lemma ([Har-Peled and Mendel, 2006])

$\mathbf{P}$: $\mathbf{n}$ points in $\mathbb{R}^{\mathbf{d}}$

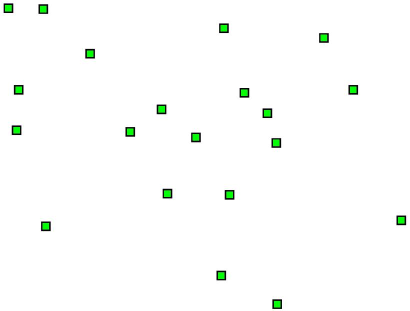
$\Longrightarrow$

(A) $\exists \mathbf{r}, \mathbf{R}$ such that $\mathbf{r} \leq \mathbf{R}$.

(B) $|\mathbf{ball}(\mathbf{p}, \mathbf{r}) \cap \mathbf{P}| \leq \mathbf{n}/\mathbf{c}$

(C) $|\mathbf{P} \setminus \mathbf{ball}(\mathbf{p}, \mathbf{R})| \leq \mathbf{n}/\mathbf{c}$

(D) $\mathbf{R} \geq (\mathbf{1} + \mathbf{1}/\mathbf{n})\mathbf{r}$.



The empty ring is “thick”.

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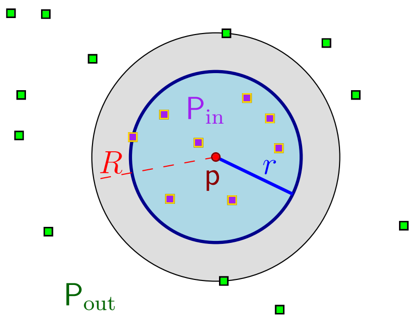
(D) $\mathbf{R} \geq (1 + 1/\mathbf{n})\mathbf{r}$.



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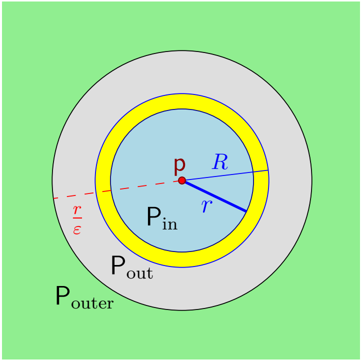
In, out, and far far away

- $\mathbf{P} = \mathbf{P}_{\text{in}} \cup \mathbf{P}_{\text{out}} \cup \mathbf{P}_{\text{outer}}$.
- $\{\mathbf{P}_{\text{in}}, \mathbf{P}_{\text{outer}}\}$: separated pair.
- Recursively separate:
  (A) $\mathbf{P}_{\text{in}}$ from itself.
  (B) $\mathbf{P}_{\text{out}} \cup \mathbf{P}_{\text{outer}}$ from itself.
- $\mathbf{P}_{\text{in}} \otimes \mathbf{P}_{\text{out}}$:
  (A) Snap $\mathbf{P}_{\text{in}} \cup \mathbf{P}_{\text{out}}$ to a grid.
  (B) Separate $\mathbf{P}_{\text{in}}^{\text{grid}}$ from $\mathbf{P}_{\text{out}}^{\text{grid}}$.



In, out, and far far away

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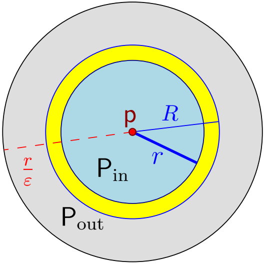
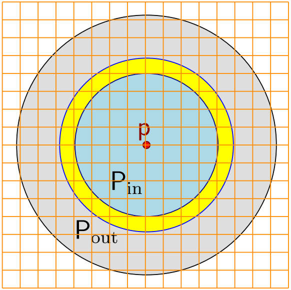
- Recursively separate:
  (A) $\mathbf{P}_{\text{in}}$ from itself.
  (B) $\mathbf{P}_{\text{out}} \cup \mathbf{P}_{\text{outer}}$ from itself.
- $\mathbf{P}_{\text{in}} \otimes \mathbf{P}_{\text{out}}$:
  (A) Snap $\mathbf{P}_{\text{in}} \cup \mathbf{P}_{\text{out}}$ to a grid.
  (B) Separate $\mathbf{P}^{\text{grid}}_{\text{in}}$ from $\mathbf{P}^{\text{grid}}_{\text{out}}$



18: SSPD construction

In, out, and far far away

- $\mathbf{P} = \mathbf{P}_{\text{in}} \cup \mathbf{P}_{\text{out}} \cup \mathbf{P}_{\text{outer}}$.
- $\{\mathbf{P}_{\text{in}}, \mathbf{P}_{\text{outer}}\}$: separated pair.
- Recursively separate:
  (A) $\mathbf{P}_{\text{in}}$ from itself.
  (B) $\mathbf{P}_{\text{out}} \cup \mathbf{P}_{\text{outer}}$ from itself.
- $\mathbf{P}_{\text{in}} \otimes \mathbf{P}_{\text{out}}$:
  (A) Snap $\mathbf{P}_{\text{in}} \cup \mathbf{P}_{\text{out}}$ to a grid.
  (B) ...bounded spread.
  (C) Separate $\mathbf{P}_{\text{in}}^{\text{grid}}$ from $\mathbf{P}_{\text{out}}^{\text{grid}}$
  (Use quadtree construction.)
  (D) Unsnap.

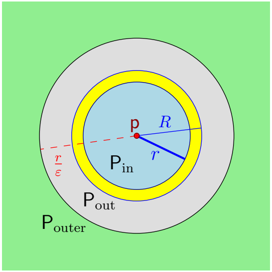


19: SSPD construction

Oh, please, not more details...

$\mathsf{P} = \mathsf{P}_{\text{in}} \cup \mathsf{P}_{\text{out}} \cup \mathsf{P}_{\text{outer}}$:

- $\{\mathsf{P}_{\text{in}}, \mathsf{P}_{\text{outer}}\}$: separated pair.
- Recursively:
  (A) P_{in} from itself.
  (B) $\mathsf{P}_{\text{out}} \cup \mathsf{P}_{\text{outer}}$ from itself.
- $\mathsf{P}_{\text{in}} \otimes \mathsf{P}_{\text{out}}$: Bounded spread case.



Max degree of a point

$\mathsf{n}_{\text{in}} = |\mathsf{P}_{\text{in}}|$, $\overline{\mathsf{n}_{\text{in}}} = |\mathsf{P}_{\text{out}}| + |\mathsf{P}_{\text{outer}}|$, and $\mathsf{n}_{\text{in}}, \overline{\mathsf{n}_{\text{in}}} \leq 0.99\mathsf{n}$.

$\mathsf{T(n)} = 1 + O\left(\varepsilon^{-d} \log n\right) + \max(\mathsf{T}(\mathsf{n}_{\text{in}}), \mathsf{T}(\overline{\mathsf{n}_{\text{in}}}))$

$$\implies \mathsf{T(n)} = O\left(\varepsilon^{-d} \log^2 n\right)$$

20: Result

Too many slides in this talk, no?

Theorem

$\mathbf{P}$: $\mathbf{n}$ points in $\mathbb{R}^{\mathbf{d}}$.
$\exists$ $\mathbf{1}/\varepsilon$-SSPD of weight $\mathbf{O}(\varepsilon^{-\mathbf{d}} \log^2 \mathbf{n})$.

See paper for the details on the improved result...

21: Conclusions

Finally this disaster is coming to an end

- New simpler constructions for SSPD.
- Works for doubling metrics.

Open problems

- More uses for ring separator?
- Spanner with small separator for doubling metrics?

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