Quasi-Polynomial Time Approximation Scheme for Sparse Subsets of Polygons

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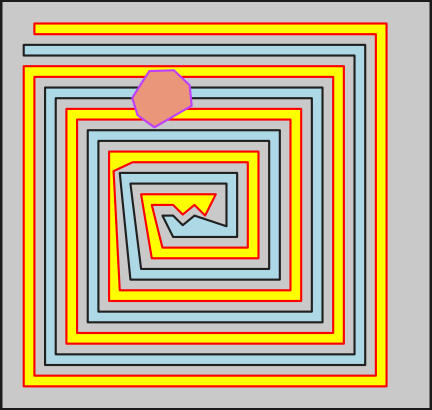
2: The problem

- $\mathcal{P}$: m polygons in the plane.
- n: Total complexity of $\mathcal{P}$.



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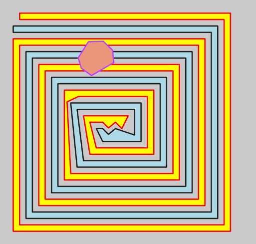

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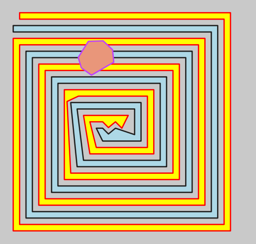

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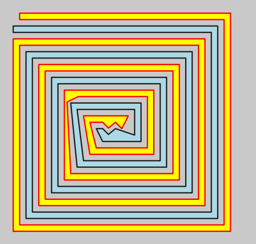
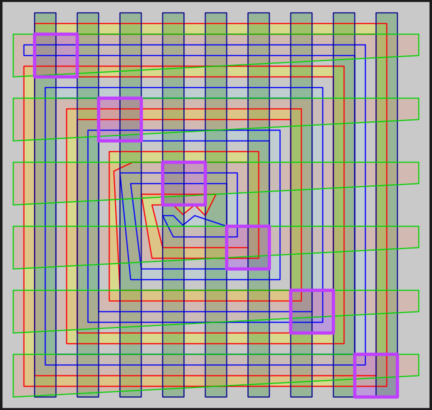


3: The problem - another example



4: The main result

Theorem

1. $\mathcal{P}$: set m simple weighted polygons.
2. n: total complexity.
3. Compute an independent set weight $\geq (1-\varepsilon)W_{\mathrm{opt}}$.
4. W_{opt}: maximum weight of optimal independent set.
5. Running time: $O\left(m^{\mathrm{poly}(\log m, 1/\varepsilon)} + m^{O(1)}n\right)$.

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Definition (PTAS)

$\varepsilon > 0$: Parameter
n: Input sizes
$(1 - \varepsilon)$-approximation
For fixed $\varepsilon > 0$, running time $n^{O(1)}$.
PTAS $=$ Polynomial time approximation scheme.

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6: Why care about QPTAS

"There is no such thing as quasicrystals, only quasi-scientists." – Linus Pauling

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Intuition

$\exists$ **QPTAS** "$\implies$" $\exists$ **PTAS**.

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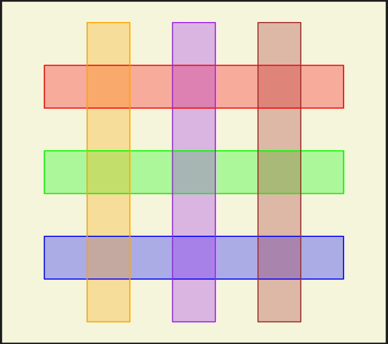
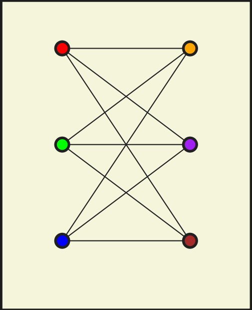
$\exists$ **QPTAS** "$\Longrightarrow$" $\exists$ **PTAS**.

7: Intersection graph

Find independent set in the graph.





8: Computing independent set

What is known...

1. **NP-Complete**.
2. No $|V|^{1-\varepsilon}$-approx if **NP** $\neq$ **ZPP** **[Hastad, 1996]**.
3. Maximum degree of graph is $\leq \mathbf{3}$: no **PTAS** possible.

9: Geometric settings

What is known...

1. PTASes for fat objects (disks/squares).
   1. [Chan, 2003, Erlebach, Jansen, and Seidel, 2005]: Quadtrees + random shifting
   2. [Chan, 2003]: Uses planar separator (unweighted).
2. Axis-parallel rectangles.
   1. [Chalermsook and Chuzhoy, 2009]: $O(\log\log n)$ approximation.
   2. [Chan and Har-Peled, 2012]: $O(\log n / \log\log n)$-approx for weighted case.
3. Pseudo-disks
   [Chan and Har-Peled, 2012]:
   $(1+\varepsilon)$-approximation (PTAS).
   Local search + separator theorem.

10: Geometric settings

Continued...

1. Line segments
   1. **[Agarwal and Mustafa, 2006]**: $O\left(\sqrt{n_{\text{opt}}}\right)$-approx.
      Dilworth's theorem.
   2. **[Fox and Pach, 2011]**: n^{ε}-approx.
      Idea: Large biclique if dense, and cheap separator if sparse.
2. New results: **[Adamaszek and Wiese, 2013, 2014]**.
   1. Good news: $(1+\varepsilon)$-approximation for rectangles.
   2. Bad news: Running time $n^{(\log n/\varepsilon)^{O(1)}}$.
      QPTAS: Quasi-polynomial time approximation scheme
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11: Why **QPTAS** are interesting?

A soup of complexity classes.

- X is **APXHard** $\implies$ no **PTAS** for X (no ε-approx).
- PCP Theorem ($\forall \varepsilon < 1/8$):
  $(1+\varepsilon)$-approx to **Max3SAT** $\implies$ solving **3SAT** exactly.
- **ETH** (**Exp. time hypothesis**):
  3SAT not solvable in $2^{o(n)}$.
- X has a **QPTAS** & X is **APXHard** $\implies$ **ETH** is false.
- X has a **QPTAS** $\implies$ X is not **APXHard**.
  ...hope for an efficient $(1+\varepsilon)$-approx.

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12: Thinking about these complexity classes

Polynomial time	$\exp(O(\log n))$
FPTAS	$\exp\left(O\left(\log \frac{n}{\varepsilon}\right)\right)$
PTAS	$\exp\left(O\left(f(\varepsilon) \log n\right)\right)$
QPTAS	$\exp\left(f(\varepsilon) \log^{O(1)} n\right)$
ETH	3SAT $\in \exp\left(n^{\Omega(1)}\right)$
EXP	$\exp\left(n^{O(1)}\right)$

Part I

Extremal graphs, Separators and Sparsity

13: A quicky on planar separators

1. **[Lipton and Tarjan, 1977]**
   Every planar graph has a separator of size $O(\sqrt{n})$.
2. **[Miller, 1986]**
   2-connected planar graph: $\exists$ **cycle** separator $O(\sqrt{n})$.
3. **[Miller, Teng, Thurston, and Vavasis, 1997]**
   Circle packing theorem $\implies$ planar separator.
4. [Har-Peled, 2013]: Circle packing easily implies planar cycle separator.
5. Circle packing theorem **[Koebe, 1936]**.

14: When are graphs sparse?

- $G = (V, E)$, $n = |V|$.
- When $|E|$ is near linear?
- Planar graphs, low genus graphs. Etc.
- If $K_{t,t}$ is a forbidden subgraph $\implies |E| \le O(n^{2-1/t})$

Theorem ([Fox and Pach, 2008])

- $\mathcal{P}$: *Set of n curves in the plane ($O(1)$ intersections for pair).*
- $G = (\mathcal{P}, E)$: *Intersection graph.*
- *For any subset $X \subseteq \mathcal{P}$: $|E(G_X)| \le |X|^{2-\delta}$. δ: Constant*
- $\implies |E(G_X)| = O(n)$.

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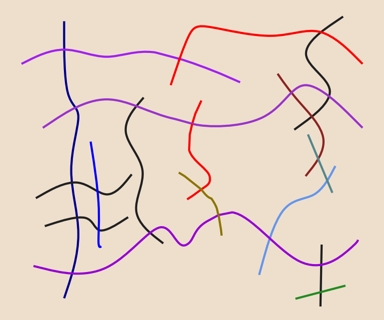

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4. Cut at S, & recurse.

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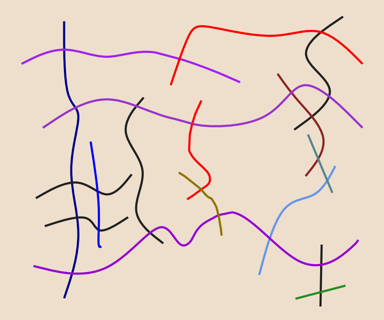

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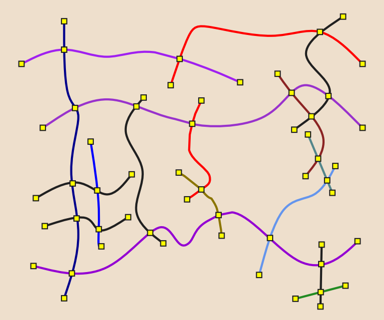

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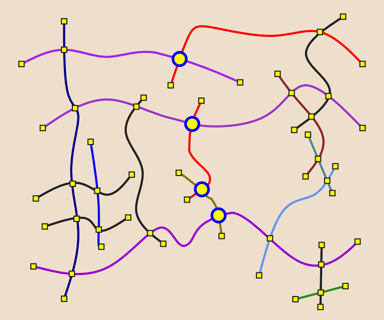

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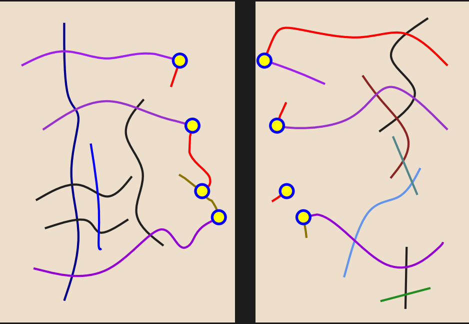

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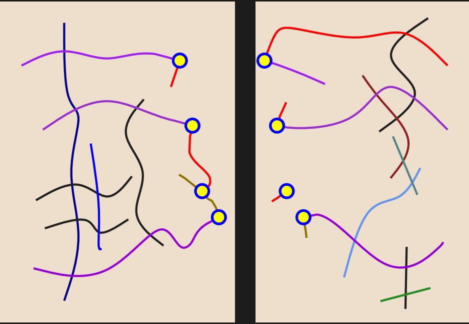

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Part II

Independent sets are sparse subsets

16: Properties of sparse subsets

- $\mathcal{P}$: Set of polygons.
- $\mathbf{\Pi}$: Property (e.g., polygons are disjoint).
- $\mathbf{\Pi}_{\mathcal{P}} = (\mathcal{P}, \mathcal{I})$, where
  $\mathcal{I} = \left\{ X \subseteq \mathcal{P} \;\middle|\; X \text{ has property } \mathbf{\Pi} \right\}$.
- **Task:** Compute heaviest $X \in \mathbf{\Pi}_{\mathcal{P}}$.
- Require of the property:
  - **hereditary**: $\forall X \in \mathbf{\Pi}_{\mathcal{P}} \quad \forall Y \subseteq X \implies Y \in \mathbf{\Pi}_{\mathcal{P}}$.
  - **mergeable**; $\forall X, Y \in \mathbf{\Pi}_{\mathcal{P}} \quad X \cap Y = \emptyset \implies X \cup Y \in \mathbf{\Pi}_{\mathcal{P}}$.
  - **sparse**: $\forall X \in \mathbf{\Pi}_{\mathcal{P}} \implies |E(G_X)| \le c' |X|^{2-\delta}$.
  - **exponential time checkable**:
    $\forall X \subseteq \mathcal{P}$, decide if $X \in \mathbf{\Pi}_{\mathcal{P}}$ in time $2^{|X|^{O(1)}}$.

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  - **mergeable**: $\forall X, Y \in \Pi_{\mathcal{P}} \quad X \cap Y = \emptyset \implies X \cup Y \in \Pi_{\mathcal{P}}$.
  - **sparse**: $\forall X \in \Pi_{\mathcal{P}} \implies |E(G_X)| \leq c' |X|^{2-\delta}$.
  - **exponential time checkable**: $\forall X \subseteq \mathcal{P}$, decide if $X \in \Pi_{\mathcal{P}}$ in time $2^{|X|^{O(1)}}$.

16: Properties of sparse subsets

1. $\mathcal{P}$: Set of polygons.
2. Π: Property (e.g., polygons are disjoint).
3. $\Pi_{\mathcal{P}} = (\mathcal{P}, \mathcal{I})$, where
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17: Main result II

Theorem

- $\mathcal{P}$: *weighted set of* m *polygons.*
- n: *total complexity*
- *every pair of polygons intersects* $O(1)$ *times.*
- $\Pi_{\mathcal{P}}$: *hereditary* $+$ *sparse* $+$ *mergeable property* $+$ *exponential time checkable.*
- W_{opt}: *maximum weight of a set in* $\Pi_{\mathcal{P}}$.
- $\implies$ **QPTAS** *to compute a subset* $X \subseteq \mathcal{P}$ $X \in \Pi_{\mathcal{P}}$, *and* $\omega(X) \geq (1 - \varepsilon) W_{\mathrm{opt}}$,

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18: Implications of Main Result II

$\mathcal{P}$: Set of polygons/pseudo-disks.
QPTAS for $(1-\varepsilon)$-approx to heaviest subset $\mathcal{O}$ of $\mathcal{P}$ s.t.

1. Pseudo-disks: every point covered $\leq c$ times by $\mathcal{O}$, c a constant.
   $c = 1$: Independent set.
2. $G_{\mathcal{O}}$ is planar.
3. $G_{\mathcal{O}}$ has low genus.
4. $G_{\mathcal{O}}$ does not contain $K_{s,t}$ as a subgraph. s and t constants.

Main point...

Independent set is just one possible sparse subset one can compute with these techniques.

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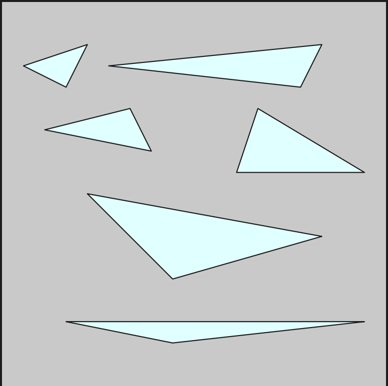
Deep in the technicalities zone

I: Simple decompositions for polygons

19: Vertical decomposition for triangles

For disjoint triangles



- m disjoint triangles.
- Rays up/down vertices. Get $O(m)$ v-traps.
- Every trapezoid defined by 4 triangles.
- **Q:** How to do VD for polygons?

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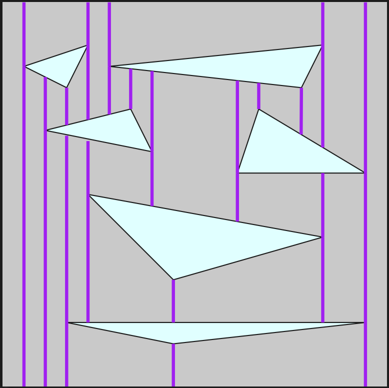

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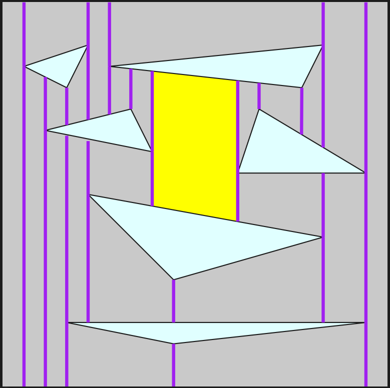

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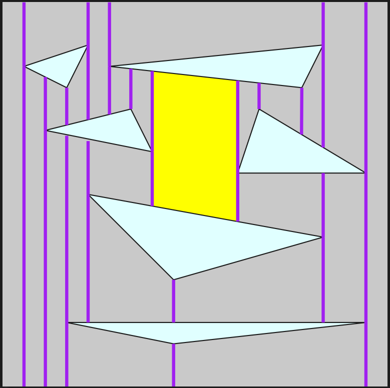

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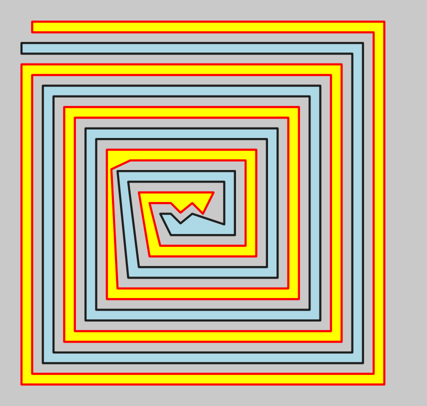

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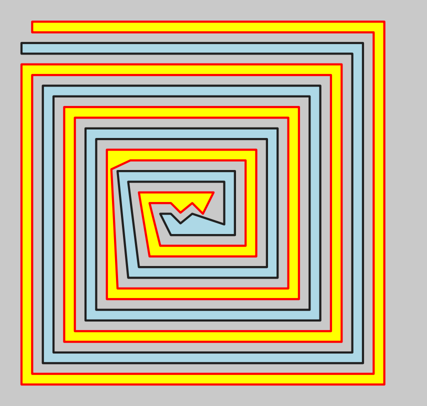

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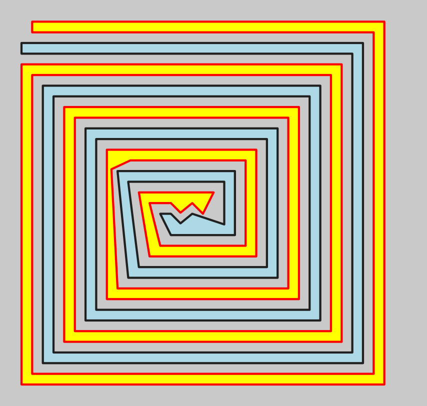

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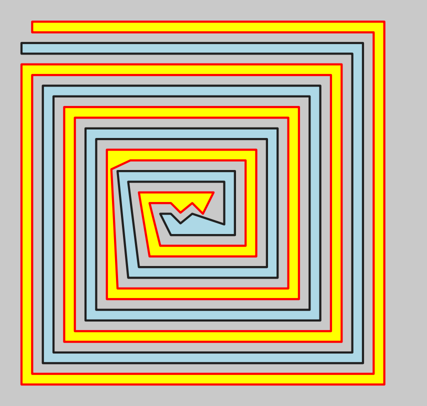

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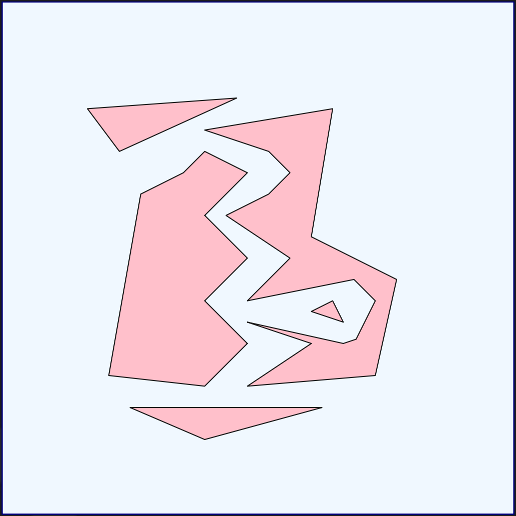
21: Corridor decomposition

Breaking into cells for polygons



- L_∞ medial axis. Deformation retract.
- Critical squares.
- Tendrils...
- ...remove them
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  - Def. by 4 polygons.
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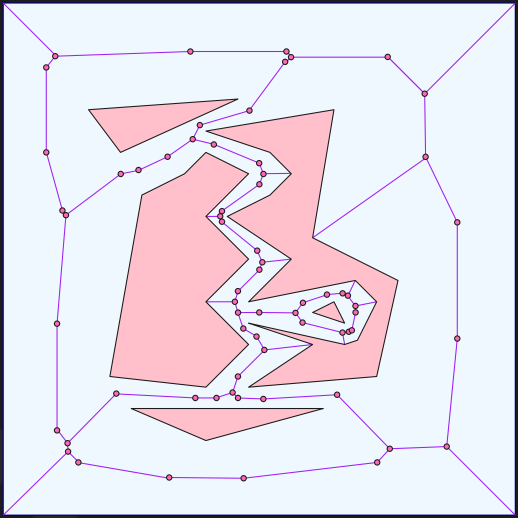

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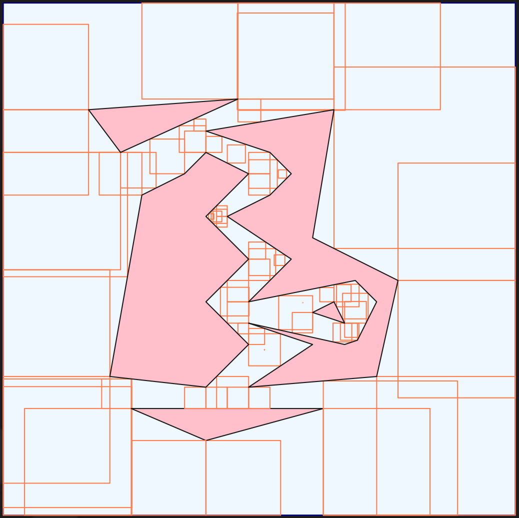

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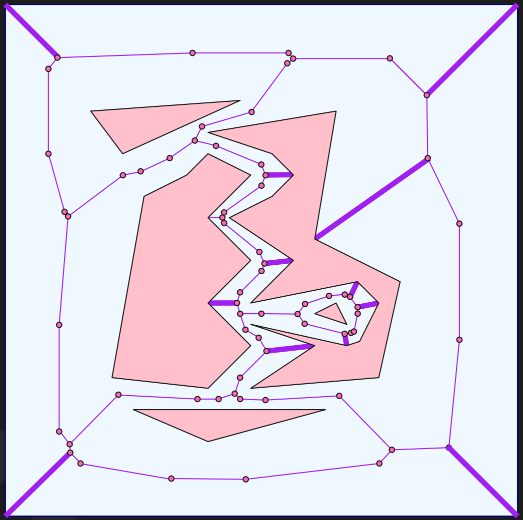

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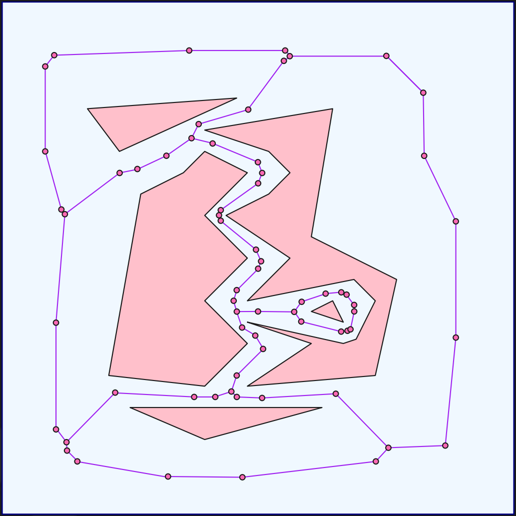

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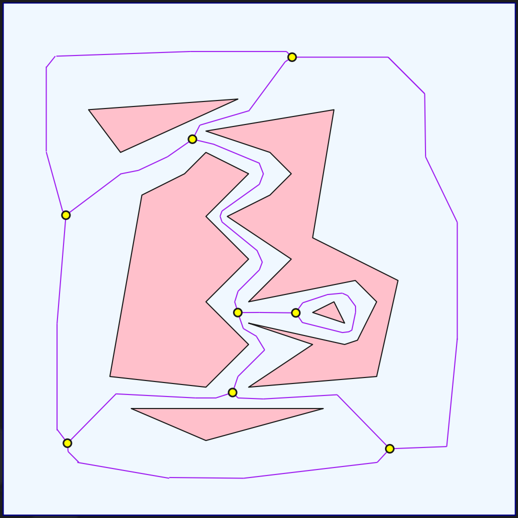

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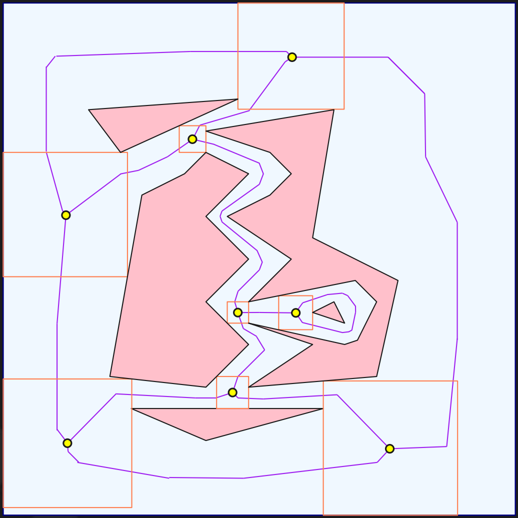

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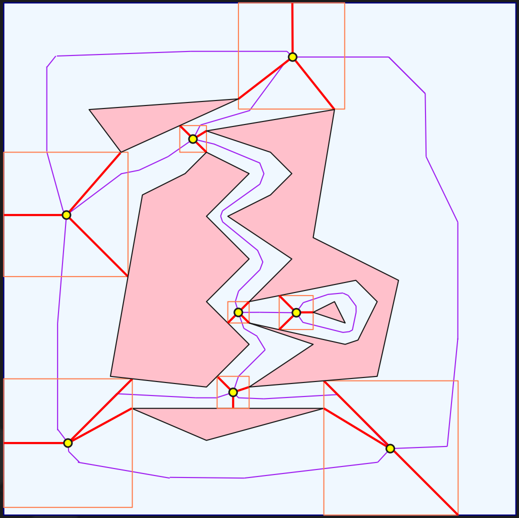

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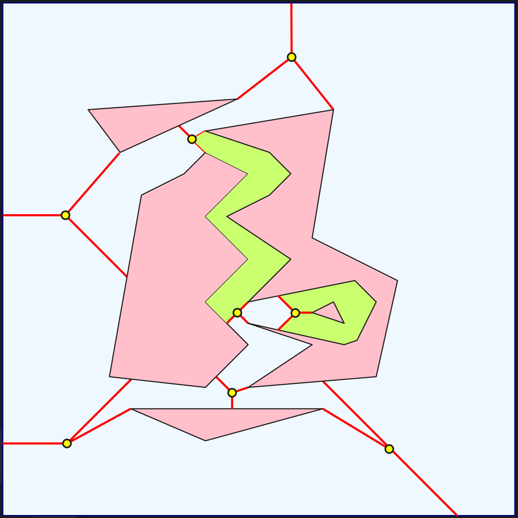

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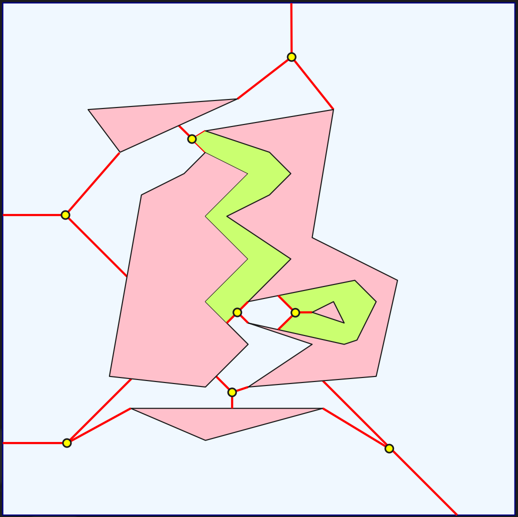

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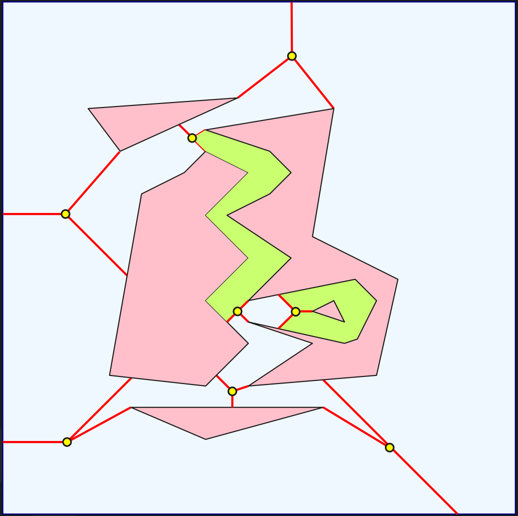

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II: Cuttings

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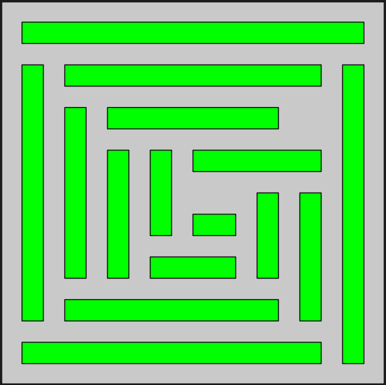

- $\mathcal{O}$: Set of n disjoint axis-parallel rectangles.
  r: Parameter
- **$1/r$-cutting**: Partition into cells, s.t. each cell intersects $\leq n/r$ rects $\mathcal{O}$.
- **[Chazelle and Friedman, 1990]**
  $1/r$-cutting with $O(r)$ cells.

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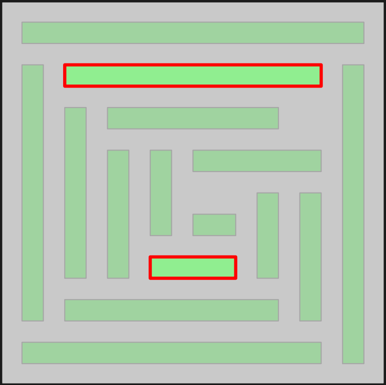

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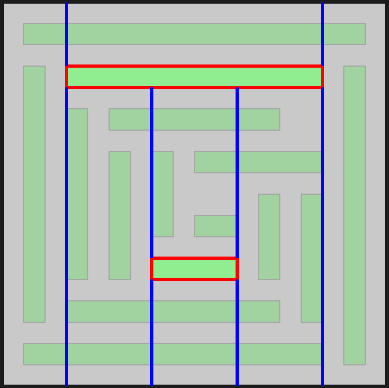

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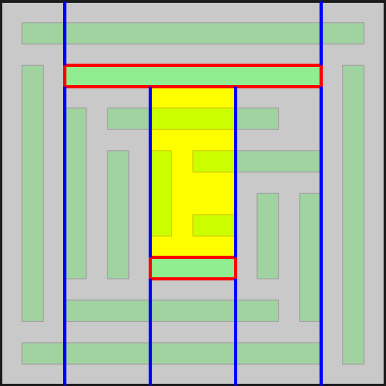

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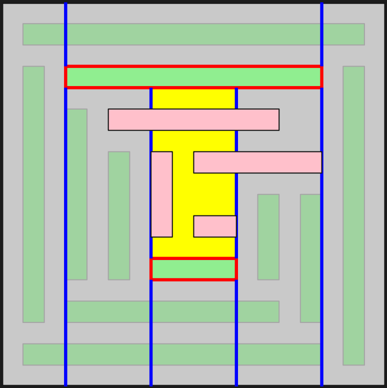

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23: Weak cuttings for polygons

Lemma

$\mathcal{P}$: disjoint weighted polygons
W: total weight W
$\implies$ *compute $1/r$-cutting with $O(r \log r)$ corridors.*

- Proof is extension of previous proof.
- Might be slightly simpler.

23: Weak cuttings for polygons

Lemma

$\mathcal{P}$: *disjoint weighted polygons*
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1. Proof is extension of previous proof.
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Part III

The algorithm

24: Recursing on optimal solution

- $\mathcal{O}$: Opt sol $m = |\mathcal{O}|$, r.
- $1/r$-cutting, with $\widetilde{O}(r)$ cells.
- Tiles form a planar graph.
- tile intersects $\leq \frac{m}{r}$ polys $\in \mathcal{O}$.
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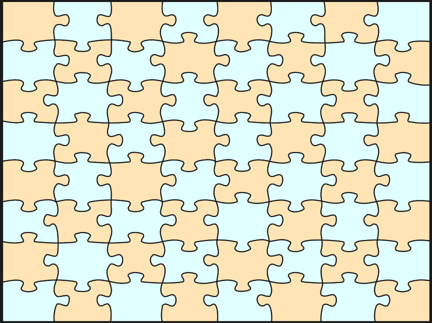
- Separator intersects $\widetilde{O}((m/r)\sqrt{r}) = \widetilde{O}(m/\sqrt{r})$.
- Recurse: In : Out

24: Recursing on optimal solution



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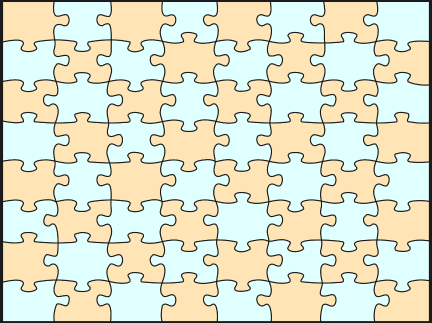

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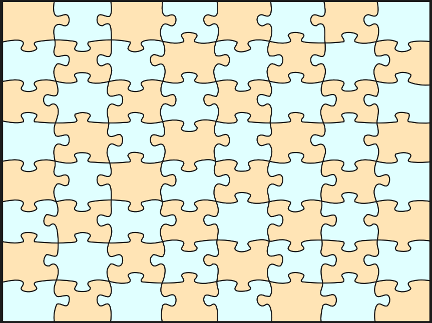

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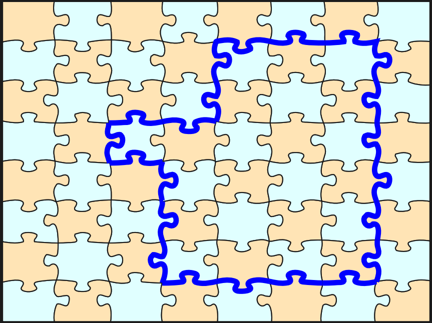

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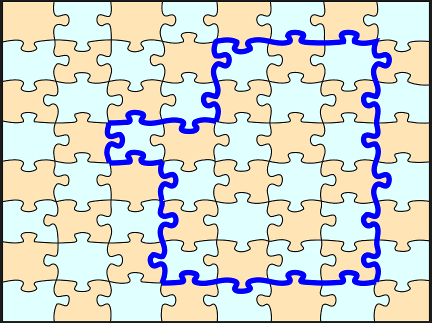

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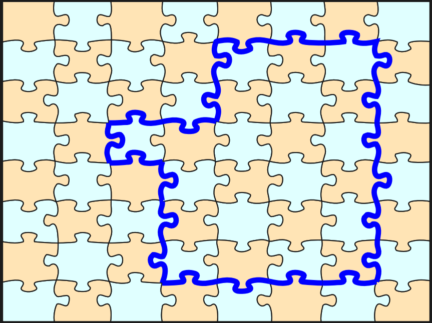

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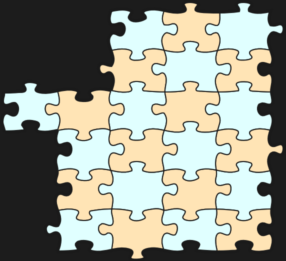

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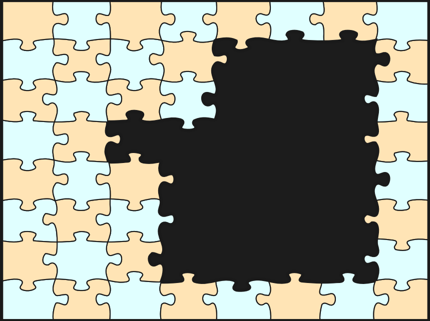

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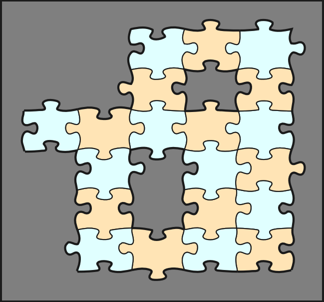
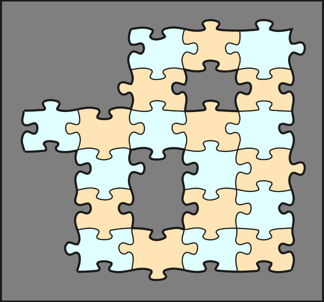
25: How a subproblem looks like



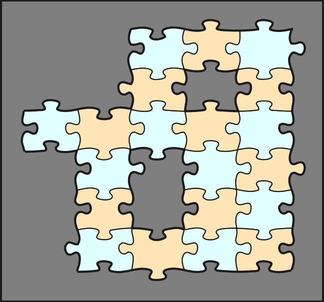
1. $\mathcal{O}$: Opt sol $|\mathcal{O}| \leq m$.
2. Subproblem defined by $O(\log m)$ cycles.
3. Each cycle has $\widetilde{O}(\sqrt{r})$ "corridor" edges.

26: Guessing the separator



- $\mathcal{O}$: Optimal solution
  r: Param (specified shortly).
- $m = |\mathcal{P}|$.
- $m^{O(1)}$ possible edges. (cuttings+corridors decomposition)
- Separator has $\widetilde{O}(\sqrt{r})$ edges.
- $m^{\widetilde{O}(\sqrt{r})}$ different subproblems in the root.
- Depth of recursion $O(\log m)$.
- $m^{\widetilde{O}(\sqrt{r}\log m)}$ different subproblems overall.

26: Guessing the separator



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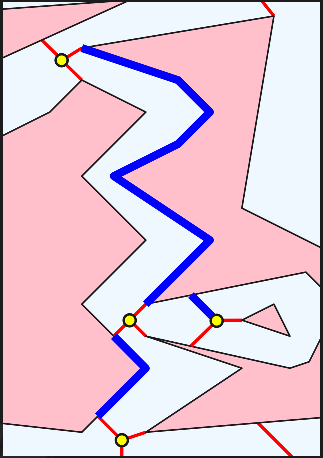

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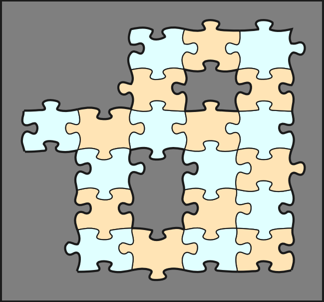

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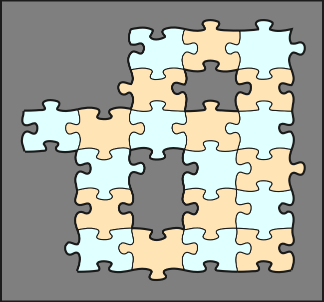

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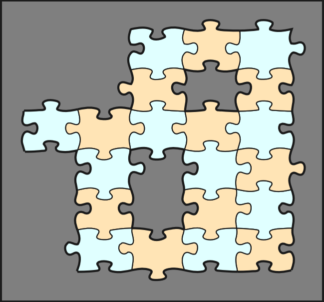

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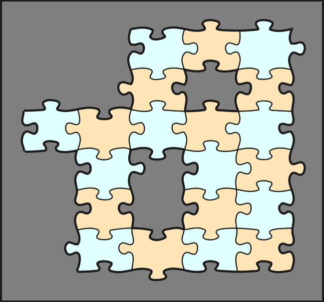

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27: What we lose in the recursion

1. $|\mathcal{O}| \leq m$.
2. Every level...
   Lose $c\sqrt{\frac{\log r}{r}}$ fraction opt sol.
3. $\leq \log m$: depth recursion.
4. Total loss: $c\sqrt{\frac{\log r}{r}} \log m \leq \varepsilon$.
5. $r = O\left(\frac{\log^3 m}{\varepsilon^2}\right) \implies (1-\varepsilon)$-approximation.
6. $m = |\mathcal{P}|$.
7. Running time $m^{\widetilde{O}(\sqrt{r} \log m)}$.
   $\implies$ Running time $m^{O\left(\left(\frac{\log m}{\varepsilon}\right)^3\right)}$ **QPTAS**.

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Part IV

Conclusions

28: Conclusions

1. Framework of [Adamaszek and Wiese, 2013, 2014].
   Insight: Using planar separator on cuttings.
2. In this paper:
   1. Extended framework...
   2. Showed **QPTAS** for independent set of polygons of arbitrary complexity.
   3. Showed techniques works for computing sparse subsets.
3. Open questions for future research:
   1. **PTAS**?
   2. Better running time.

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