Approximation Algorithms for Polynomial-Expansion and Low-Density Graphs

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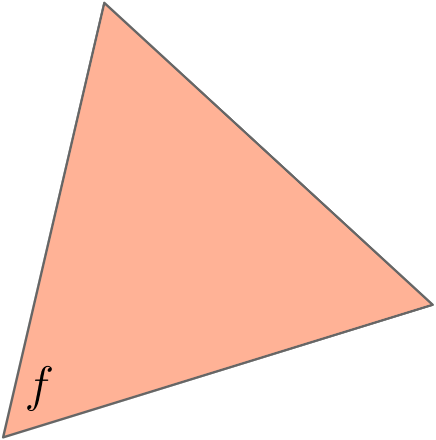
University of Illinois at Urbana-Champaign

September 27, 2015

Fat objects



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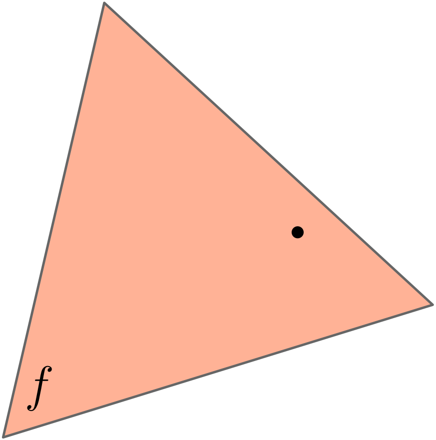

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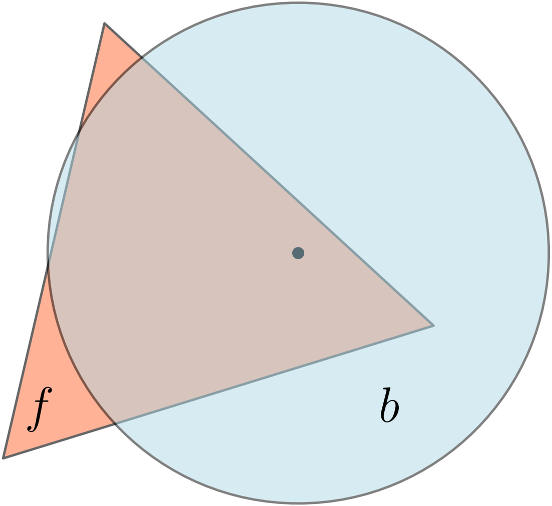

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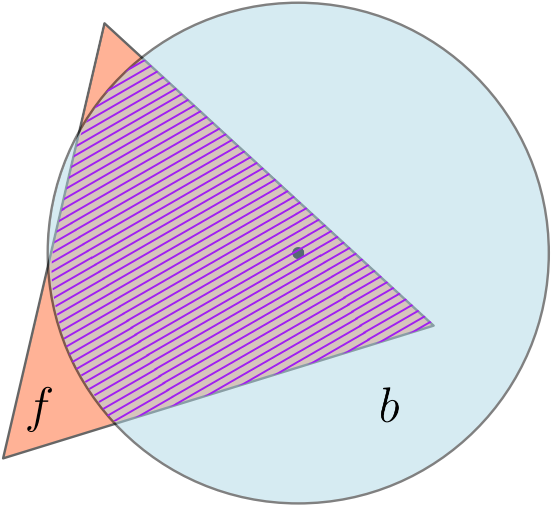

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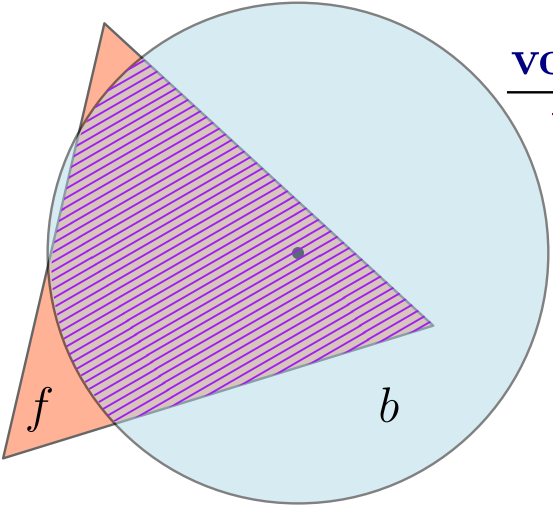
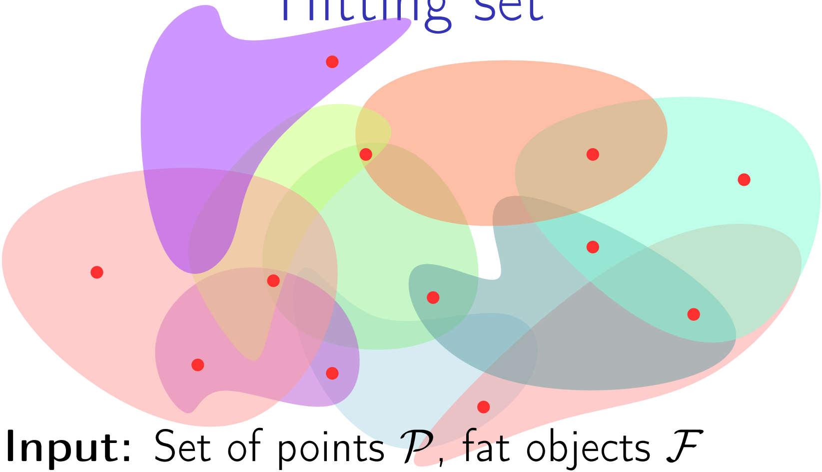


$$\frac{\mathbf{vol}(b \cap f)}{\mathbf{vol}(b)} = \Omega(1)$$

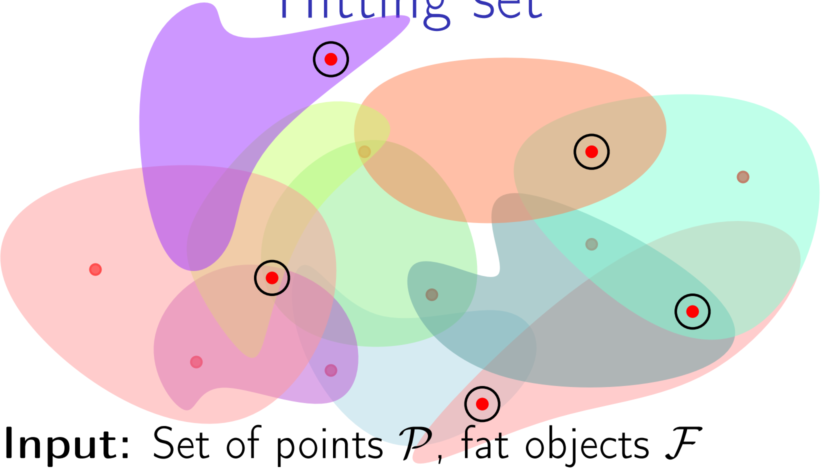
Hitting set



Input: Set of points $\mathcal{P}$, fat objects $\mathcal{F}$

Output: The smallest cardinality subset of $\mathcal{P}$ that pierces every object in $\mathcal{F}$.

Hitting set



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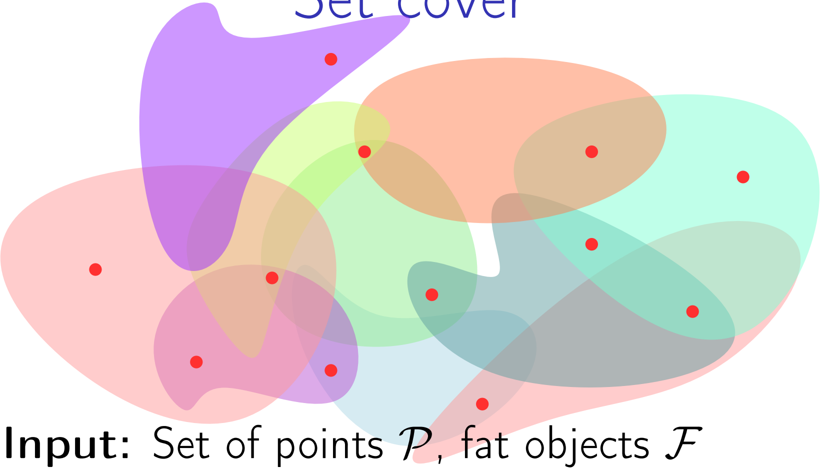
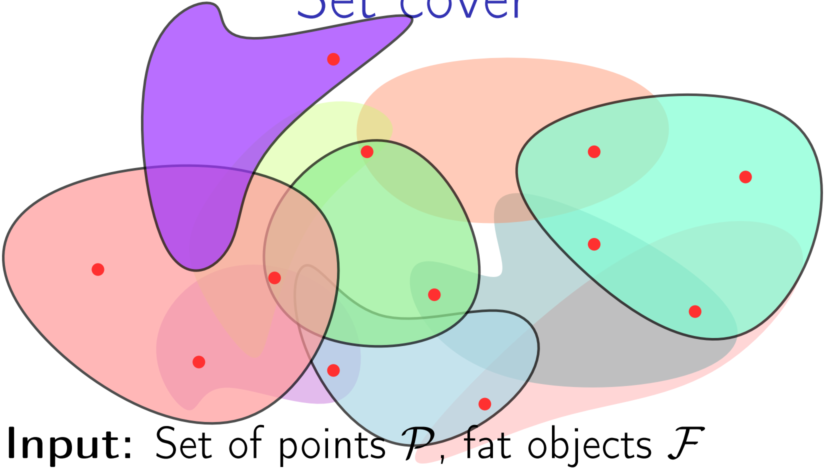
Set cover



Input: Set of points $\mathcal{P}$, fat objects $\mathcal{F}$

Output: Find the smallest cardinality subset of $\mathcal{F}$ that covers every point in $\mathcal{P}$.

Set cover



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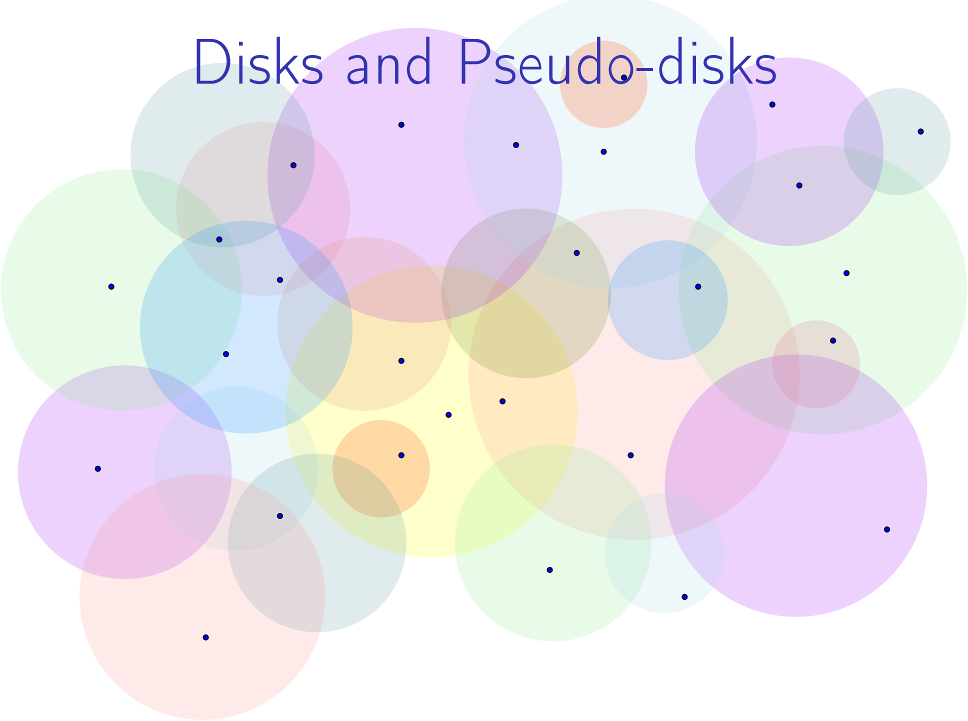
Disks and Pseudo-disks



Disks and Pseudo-disks

Set Cover

- NP-Hard [Feder and Greene, 1988]
- PTAS [Mustafa, Raman, and Ray, 2014]

Hitting Set

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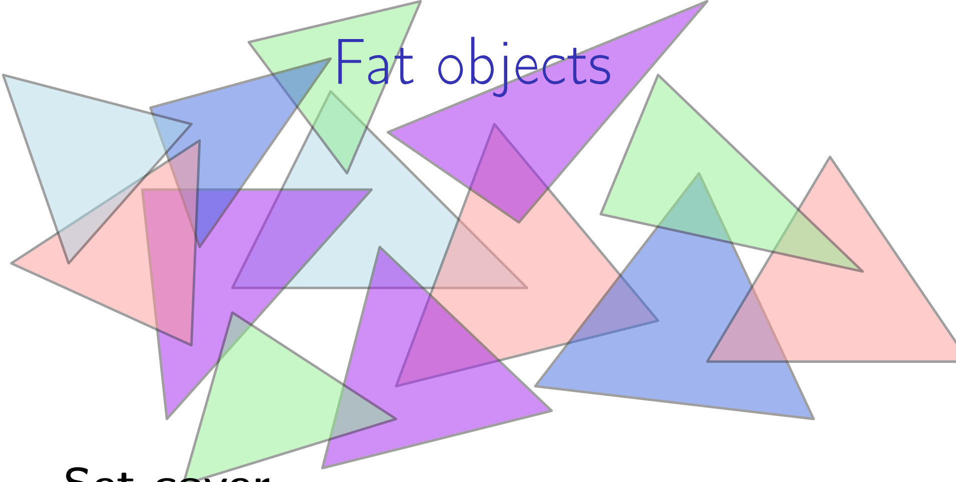
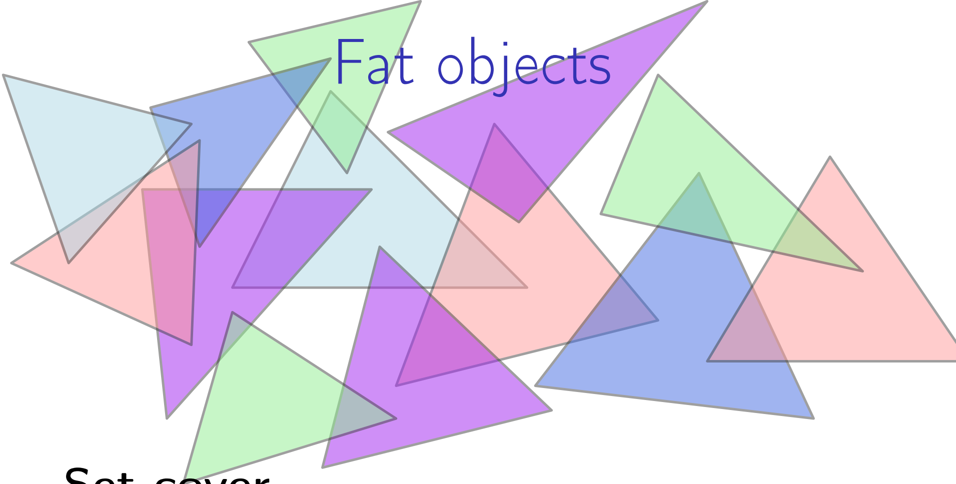
Fat objects



Set cover

$O(1)$ approx. for fat triangles of same size
[Clarkson and Varadarajan, 2007]

Fat objects

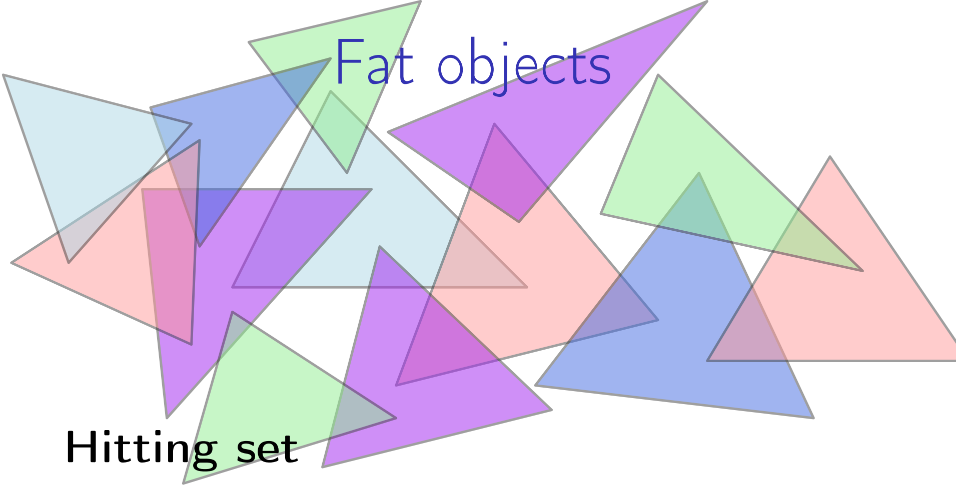


Set cover

$O(\log^* \mathrm{OPT})$ for fat objects in $\mathbb{R}^2$

[Aronov, de Berg, Ezra, and Sharir, 2014]

Fat objects



Hitting set

$O(\log\log \mathrm{OPT})$ for fat triangles of similar size

[Aronov, Ezra, and Sharir, 2010]

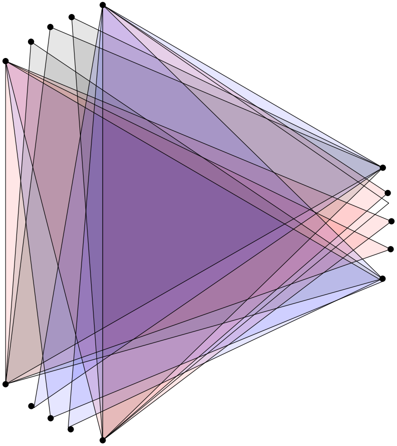
Fat, nearly equilateral triangles

Set cover

- APX-Hard

Hitting set

- APX-Hard



Fat, nearly equilateral triangles

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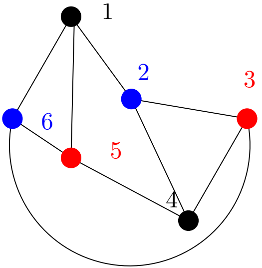

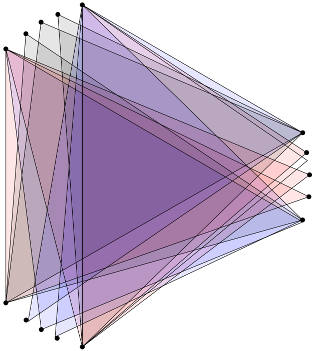

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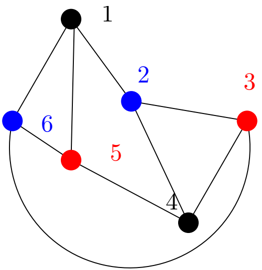

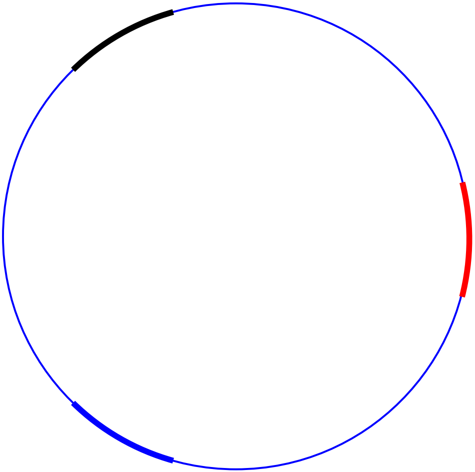

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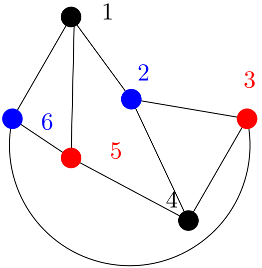

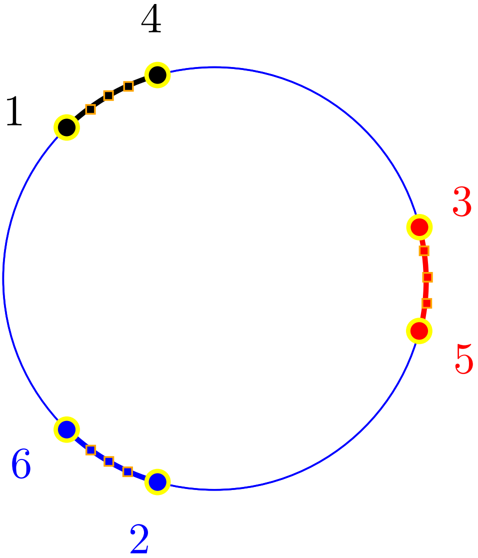

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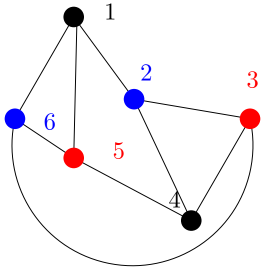

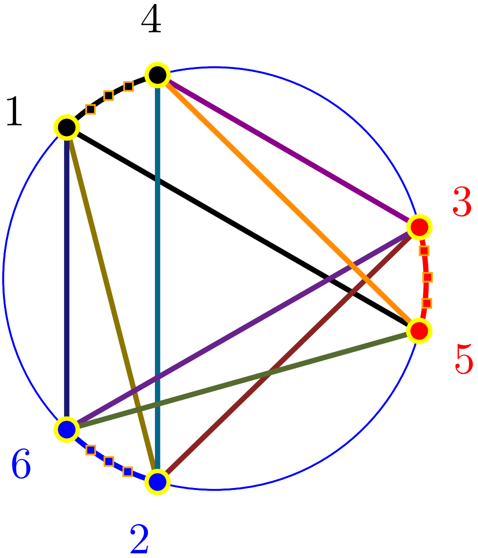

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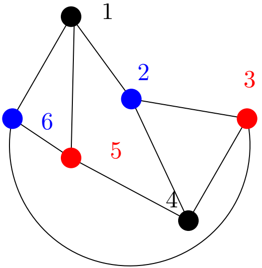

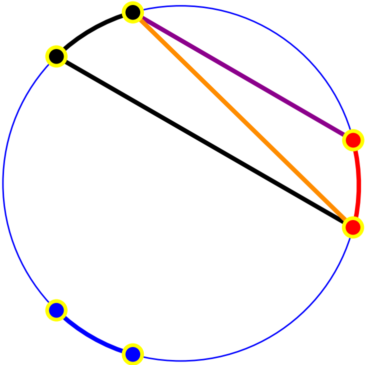

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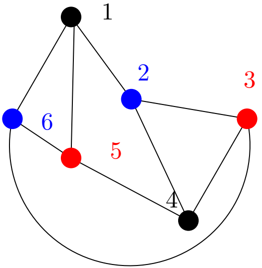

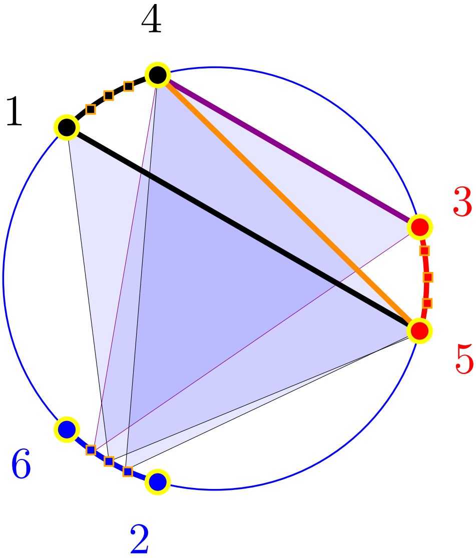

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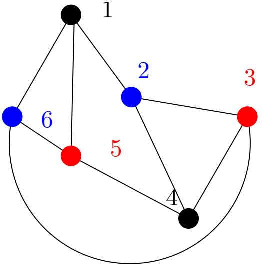

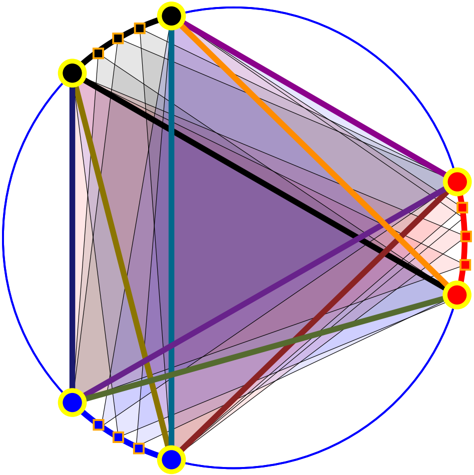

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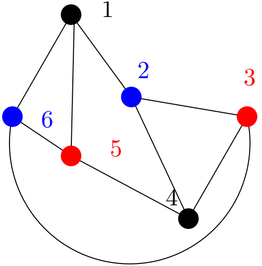
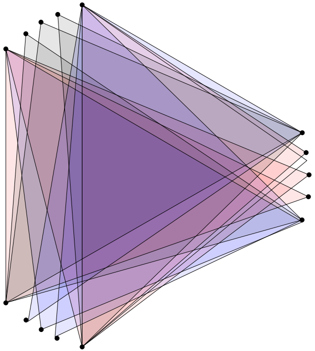
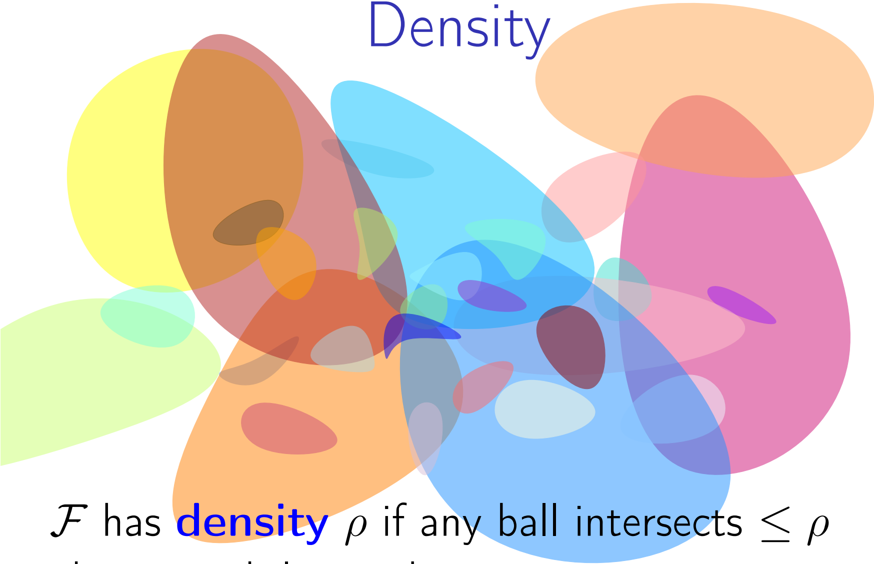




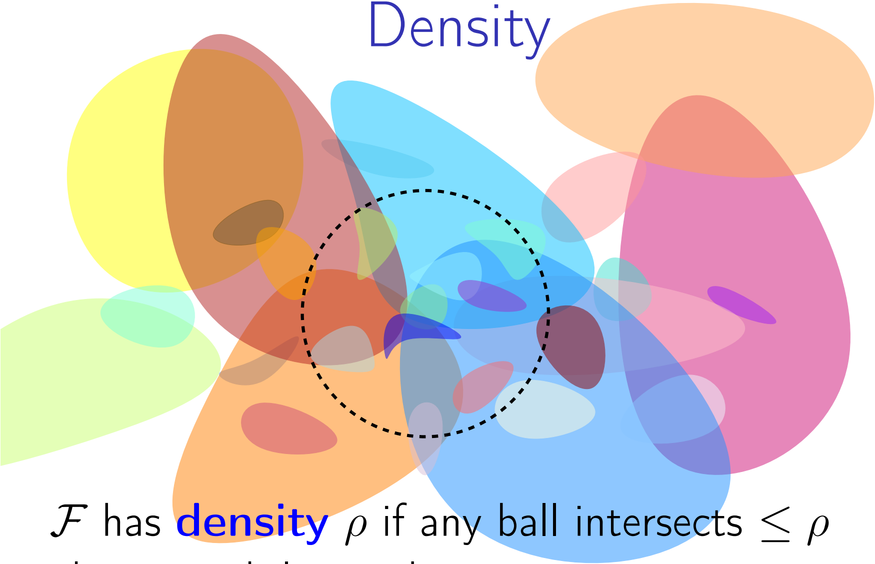
Density



$\mathcal{F}$ has **density** ρ if any ball intersects $\leq \rho$ objects with larger diameter.

[Stappen, Overmars, Berg, and Vleugels, 1998]

Density



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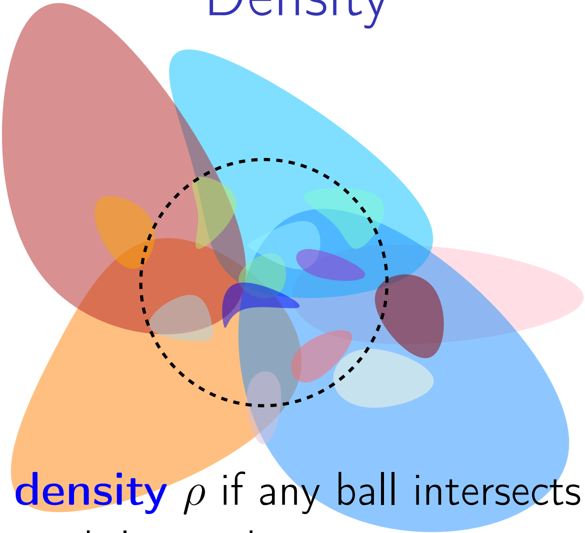

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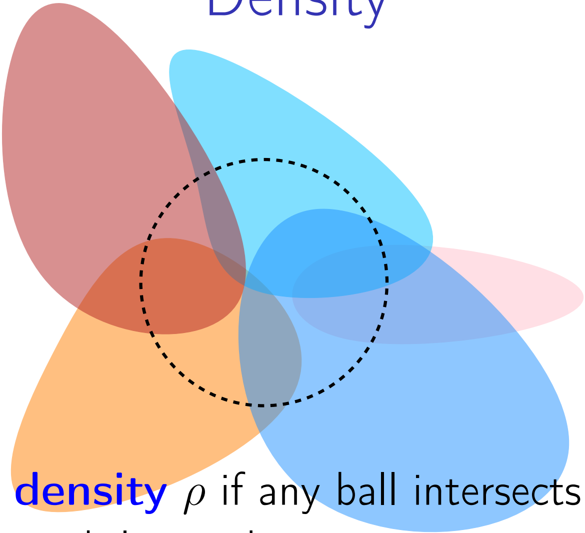

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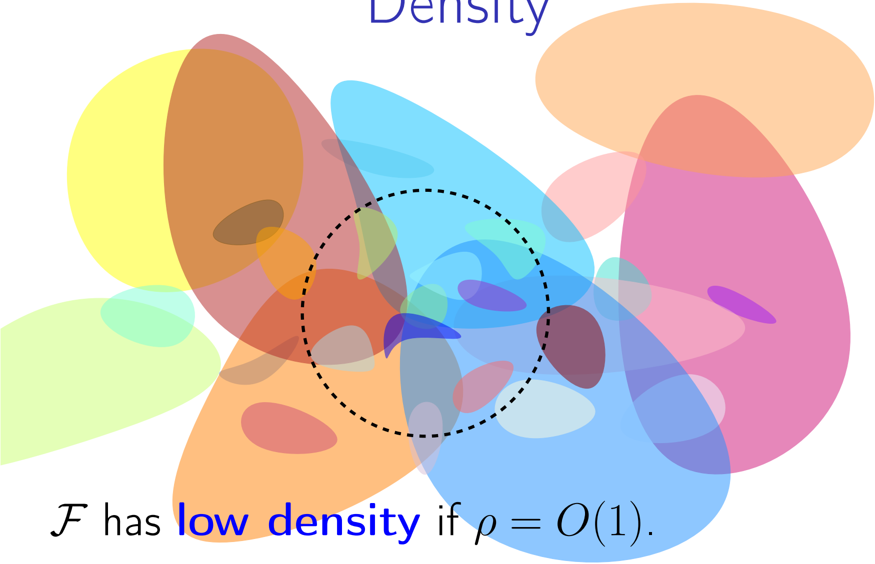
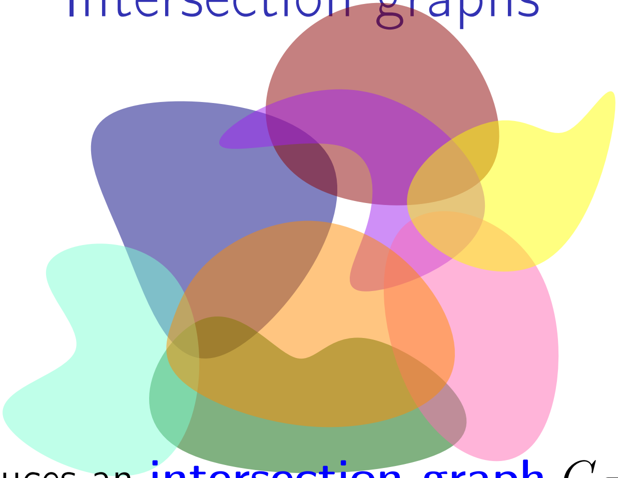


$\mathcal{F}$ has **low density** if $\rho = O(1)$.

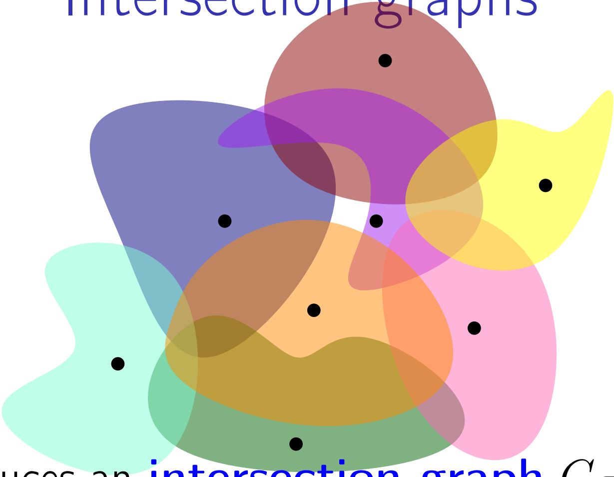
[Stappen, Overmars, Berg, and Vleugels, 1998]

Intersection graphs



$\mathcal{F}$ induces an **intersection graph** $G_{\mathcal{F}}$ with objects as vertices and edges representing overlap.

Intersection graphs



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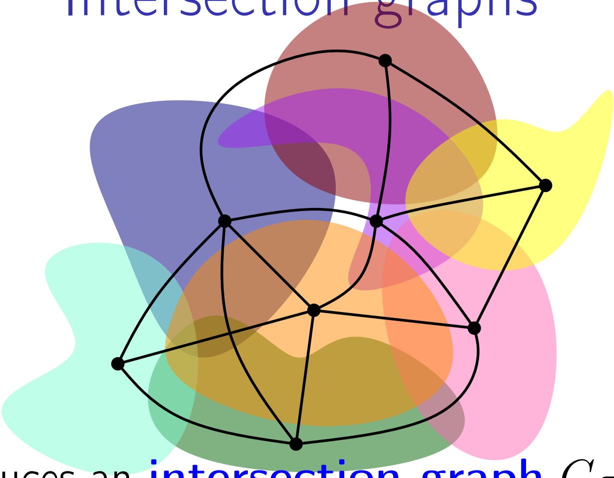

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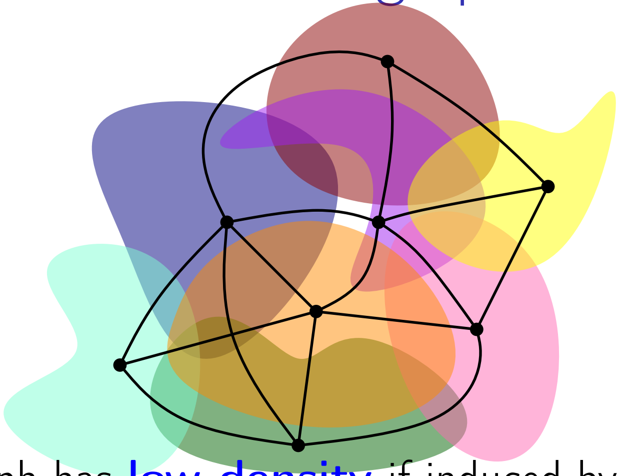
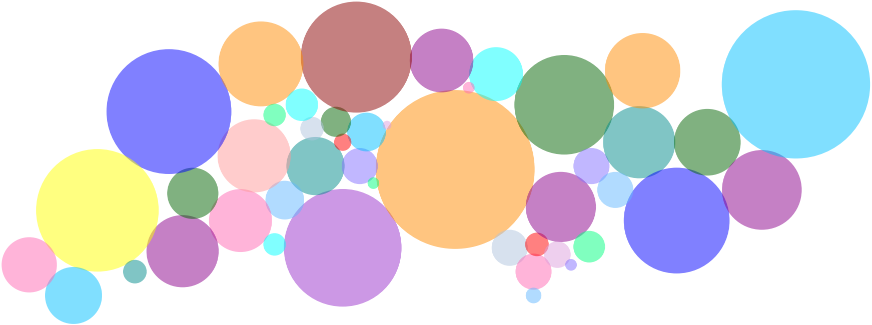


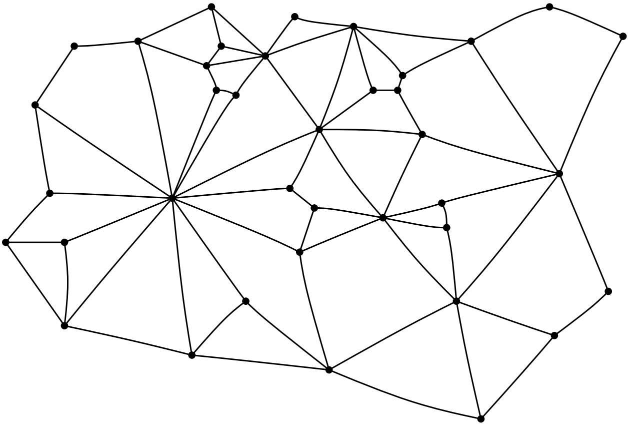
A graph has **low-density** if induced by low-density objects.

Examples of low density



Interior disjoint disks have $O(1)$ density.

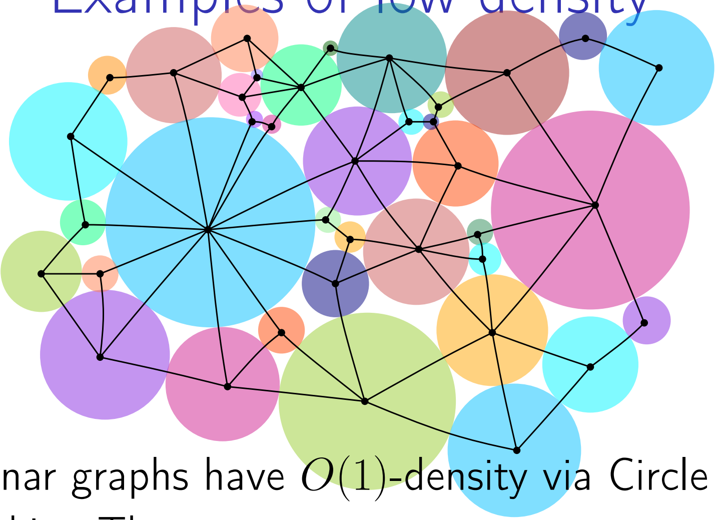
Examples of low density



Planar graphs have $O(1)$-density via Circle Packing Theorem.

[Koebe, 1936], [Andreev, 1970], [Thurston, 1985]

Examples of low density



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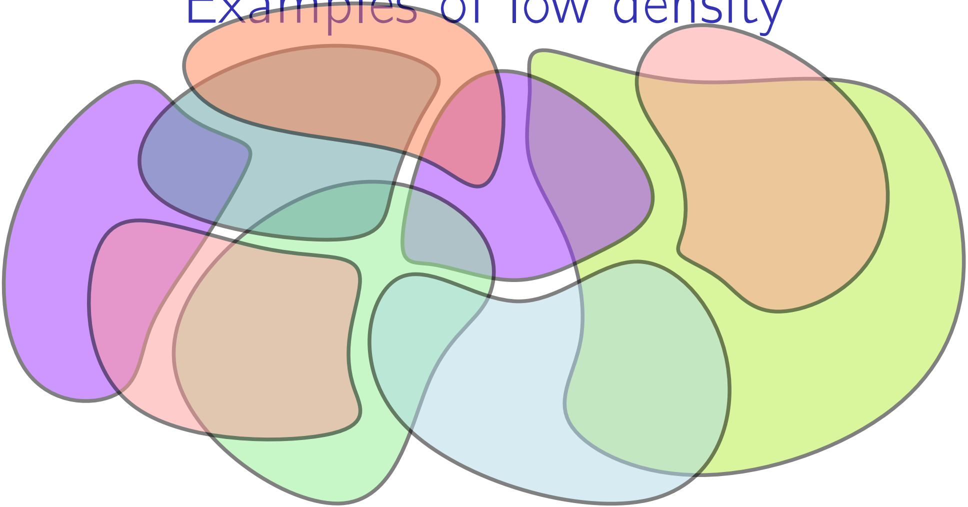
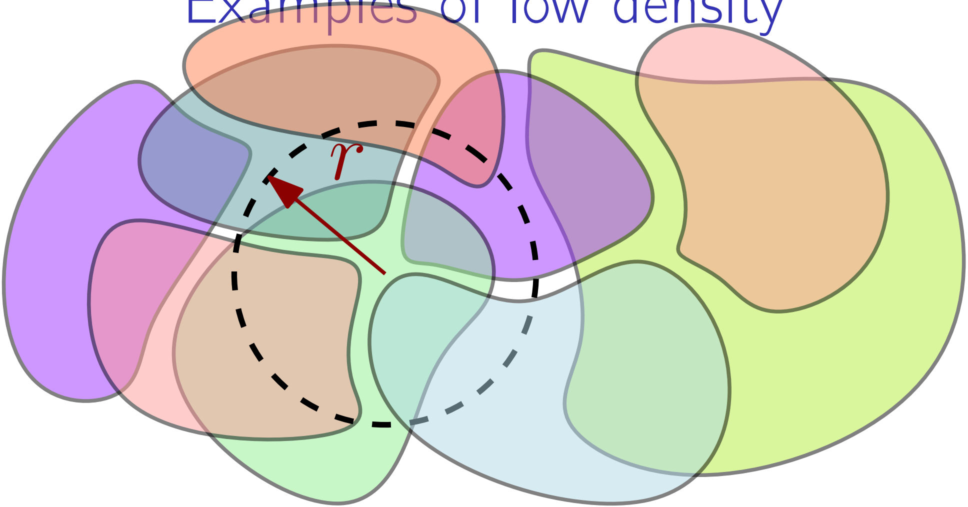


Fat convex objects in $\mathbb{R}^d$ with depth k have density $O(k2^d)$.

Examples of low density



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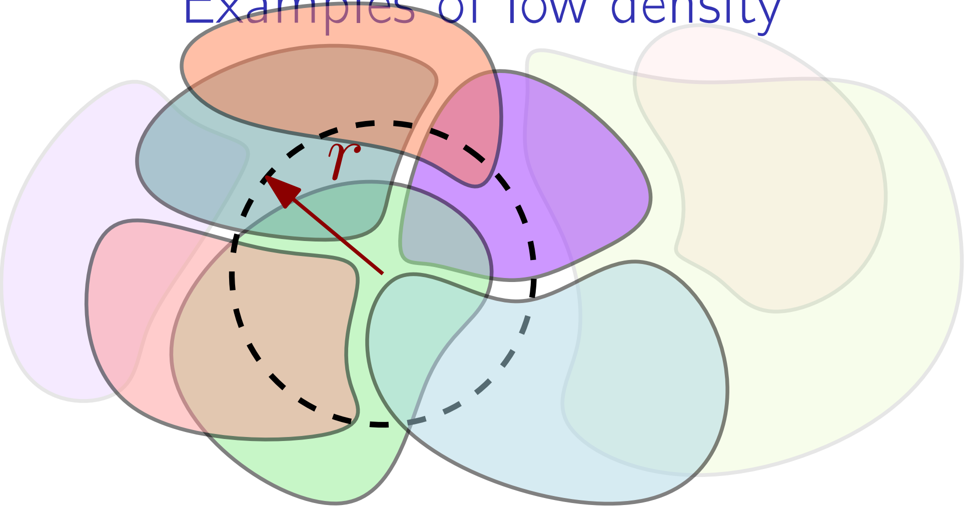
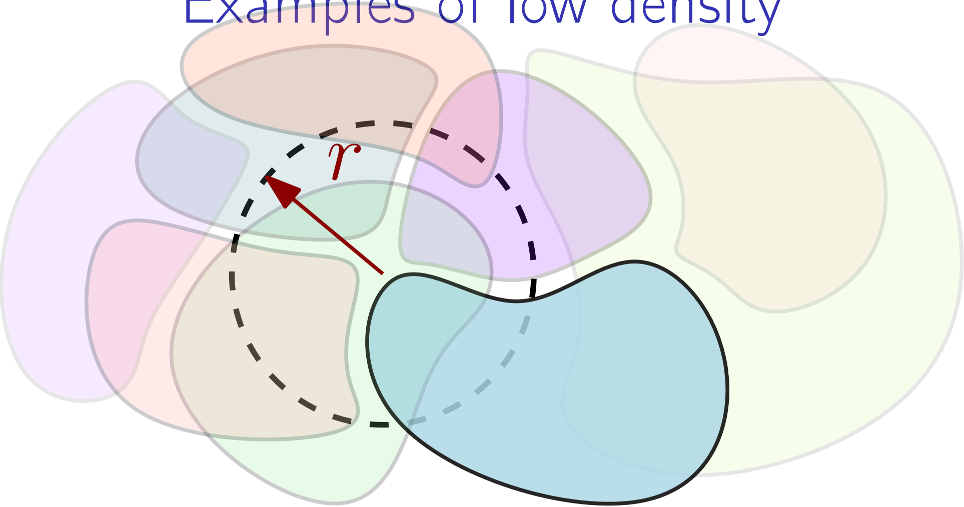


Fat convex objects in $\mathbb{R}^d$ *with depth* k *have density* $O(k2^d)$.

Examples of low density



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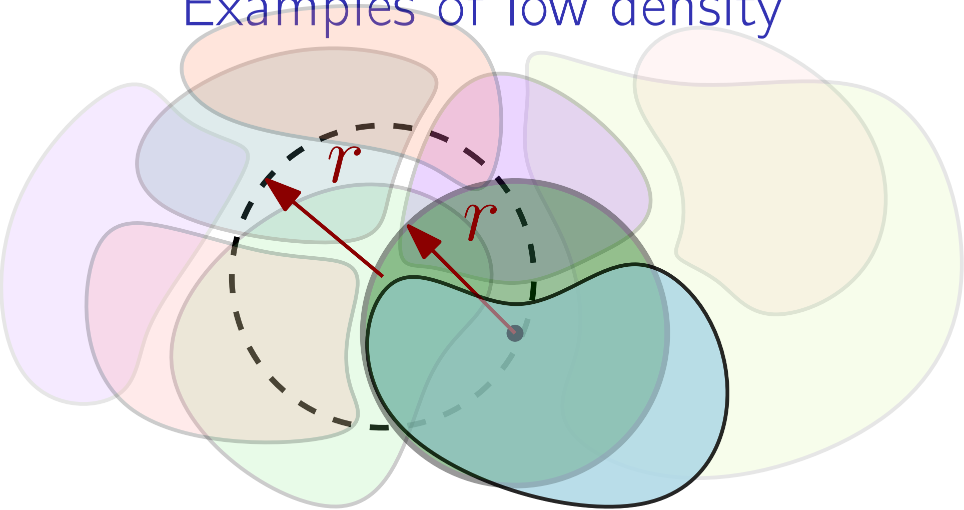

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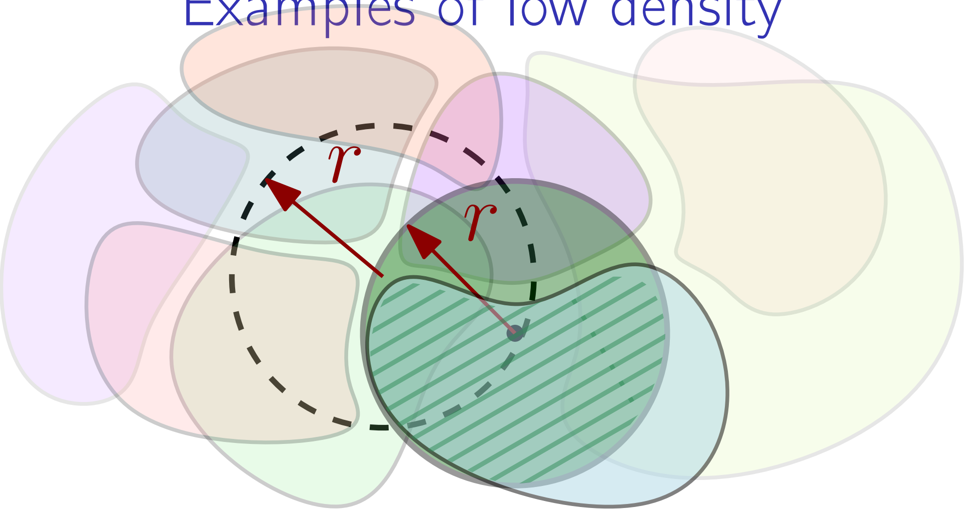

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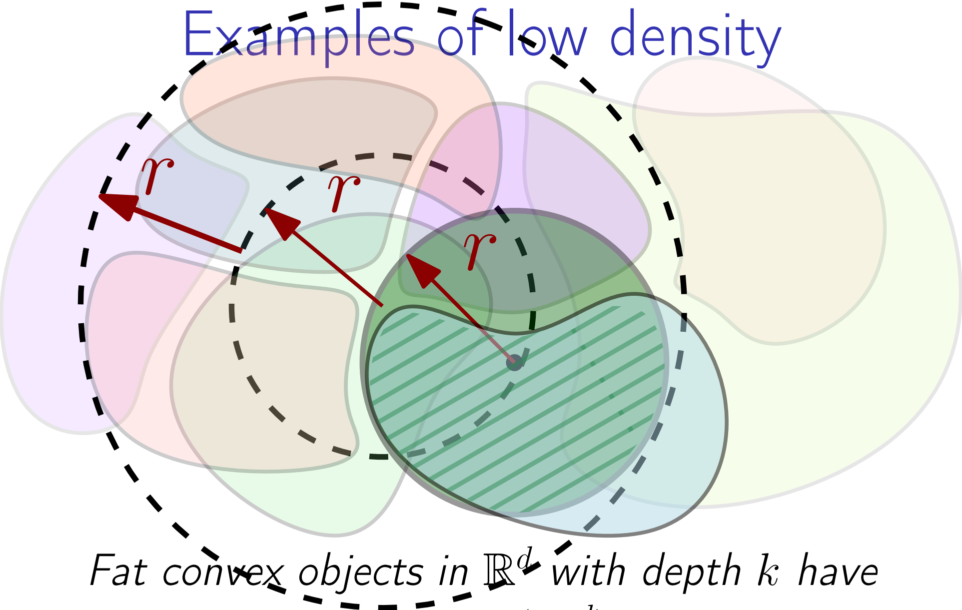
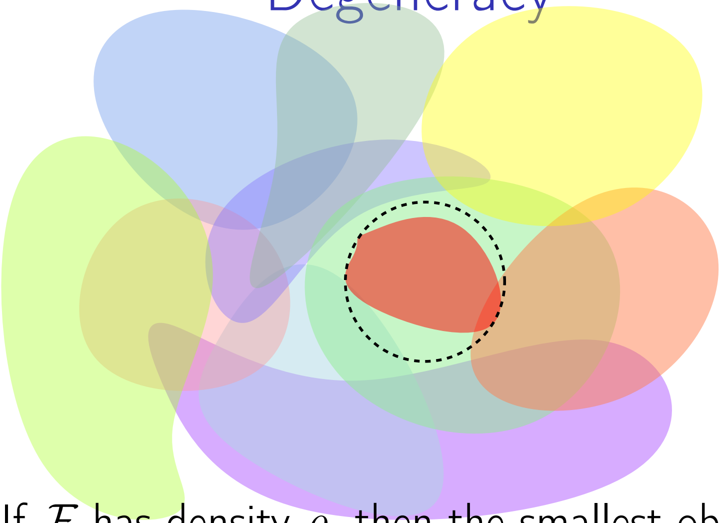


Fat convex objects in $\mathbb{R}^d$ *with depth* k *have density* $O(k2^d)$.

Degeneracy

If $\mathcal{F}$ has density ρ, then the smallest object intersects at most $\rho - 1$ other objects.

Degeneracy



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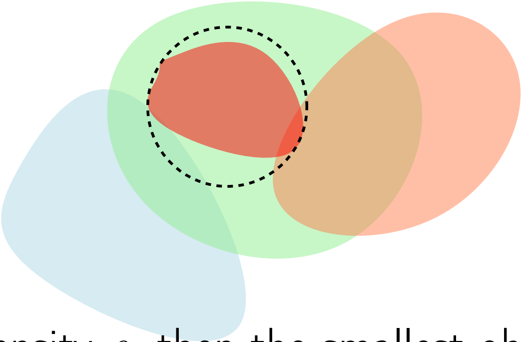
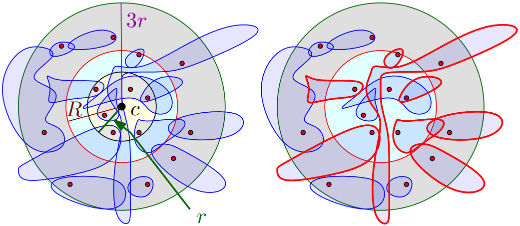


If $\mathcal{F}$ has density ρ, then the smallest object intersects at most $\rho - 1$ other objects.

Separators

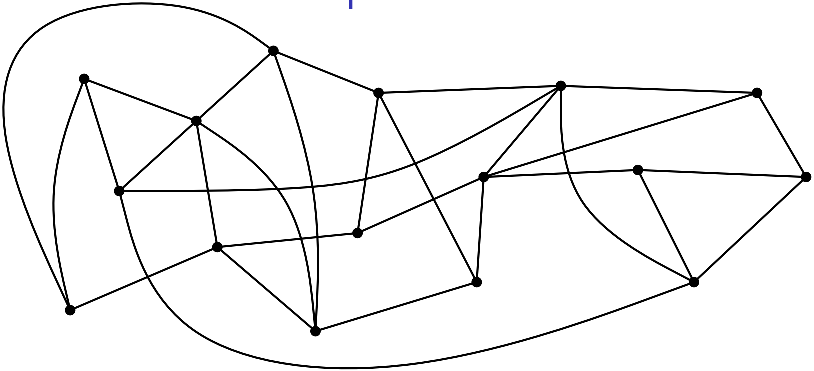
For $k \leq |\mathcal{F}|$, compute a sphere S that

- Strictly contains at least $k - o(k)$ objects and at most k objects.
- Intersects $O(\rho + \rho^{1/d} k^{1-1/d})$ objects.



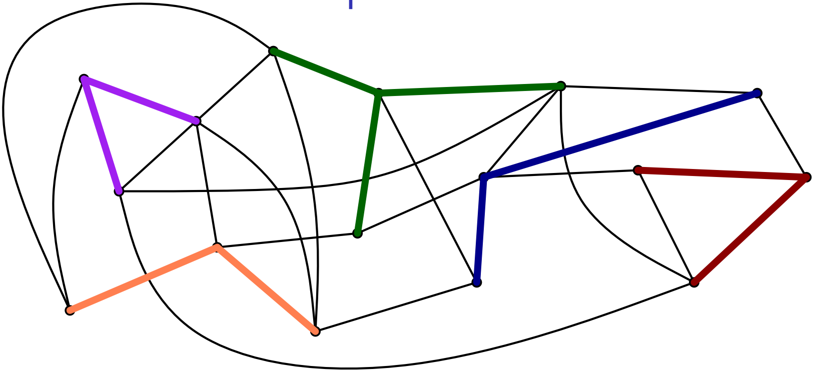
[Miller, Teng, Thurston, and Vavasis, 1997], [Smith and Wormald, 1998], [Chan, 2003]

Graph minors



A **minor** of G is a graph H obtained by contracting edges, deleting edges, and deleting vertices.

Graph minors



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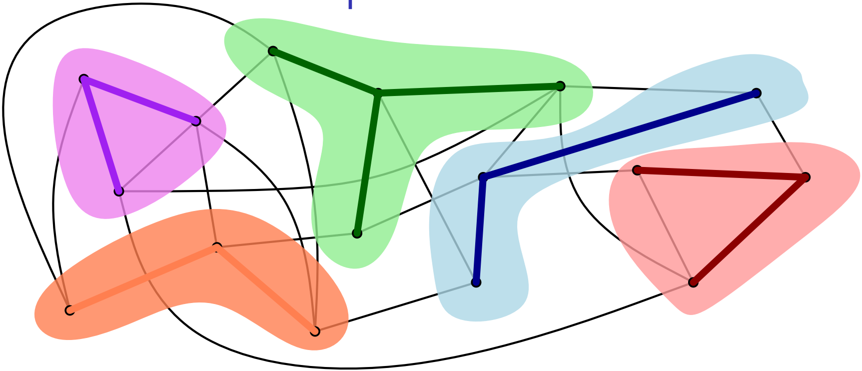

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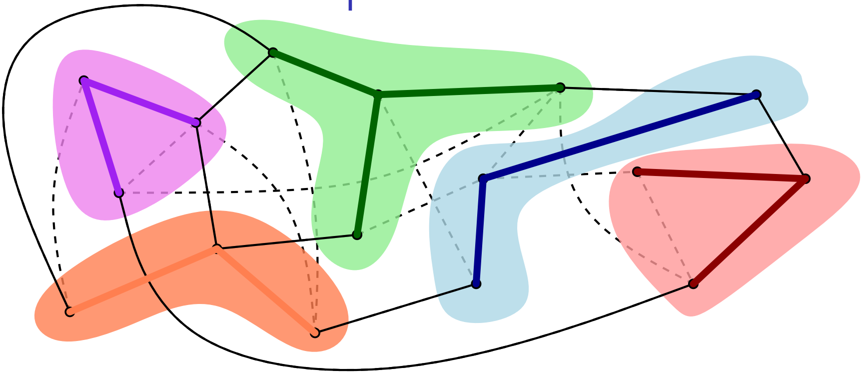

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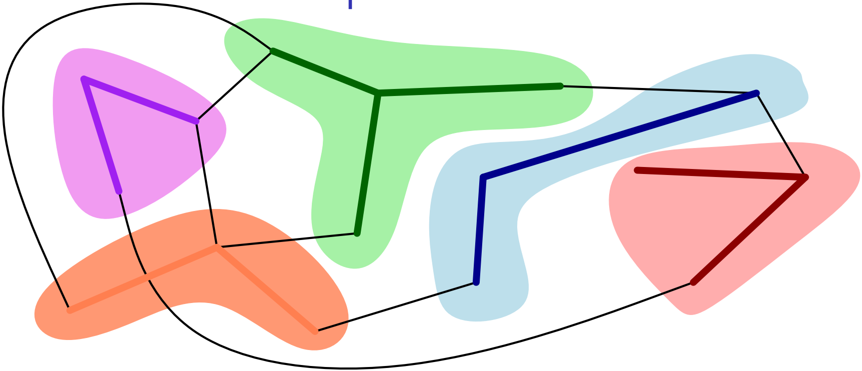

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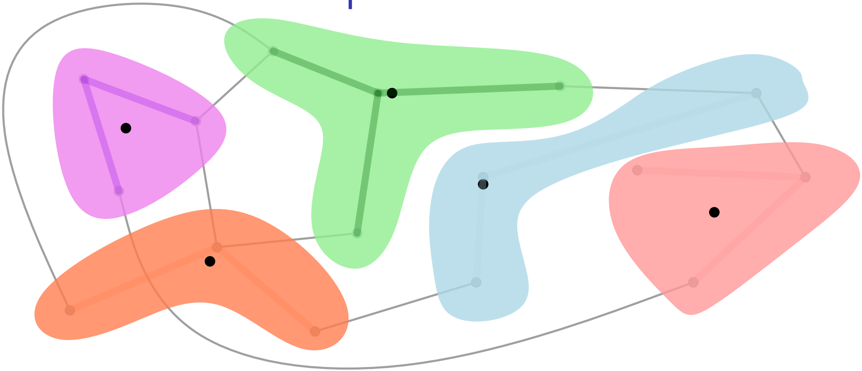

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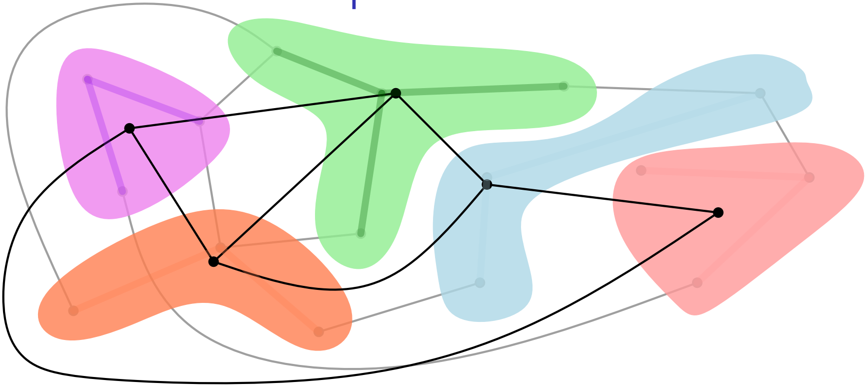

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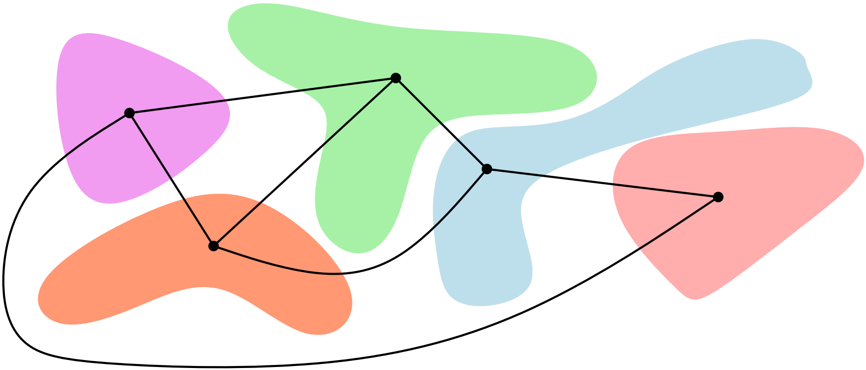
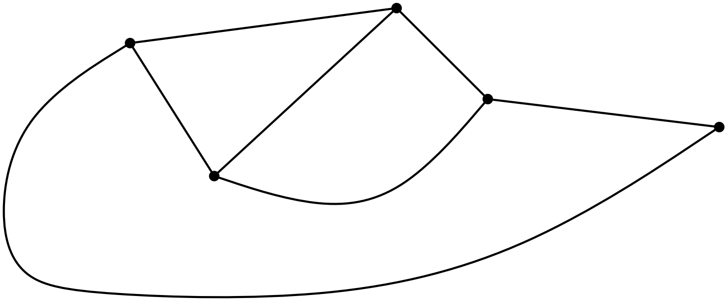


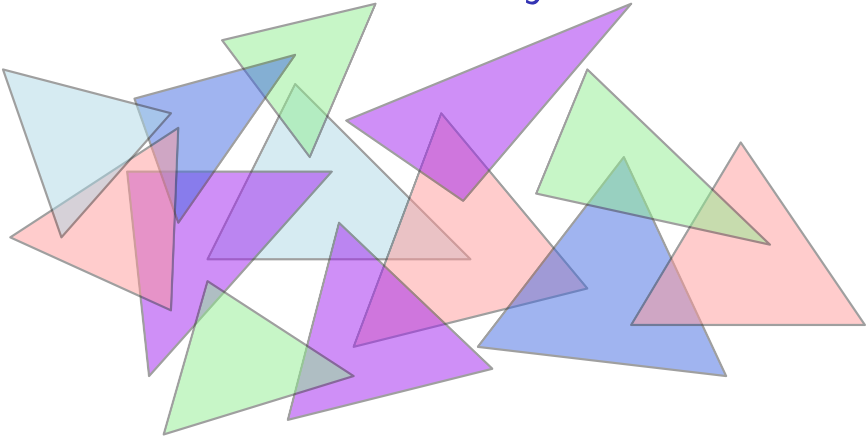
A **minor** of G is a graph H obtained by contracting edges, deleting edges, and deleting vertices.

Graph minors



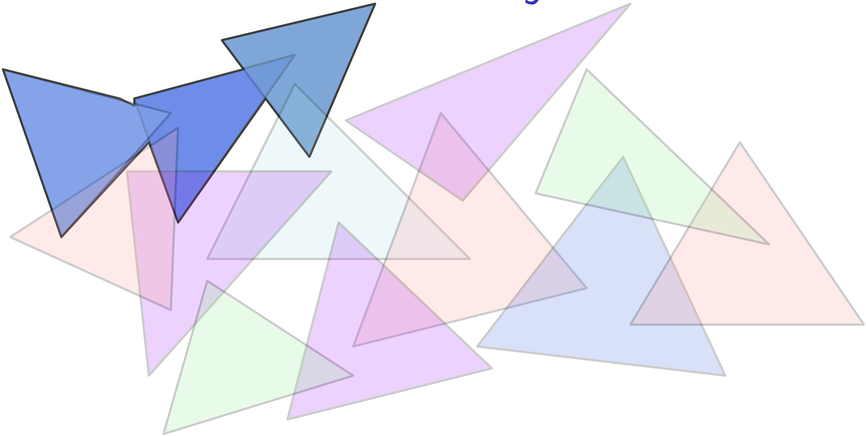
A **minor** of G is a graph H obtained by contracting edges, deleting edges, and deleting vertices.

Minors of objects



$\mathcal{G}$ is a **minor** of $\mathcal{F}$ if it can be obtained by deleting objects and taking unions of overlapping of objects.

Minors of objects



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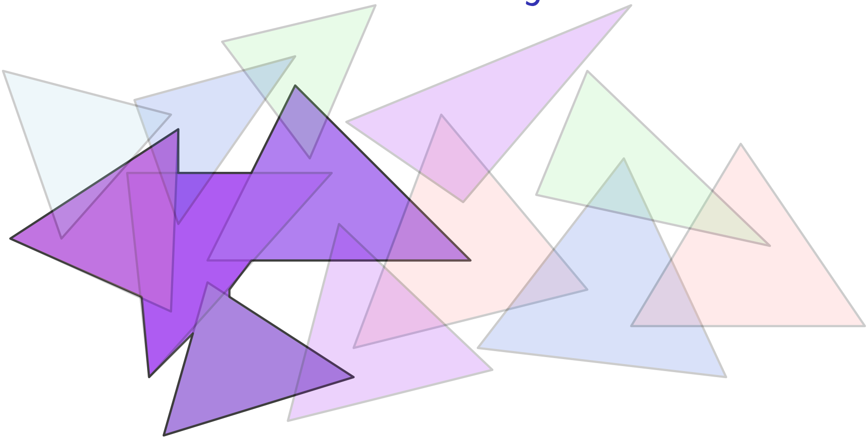
Minors of objects



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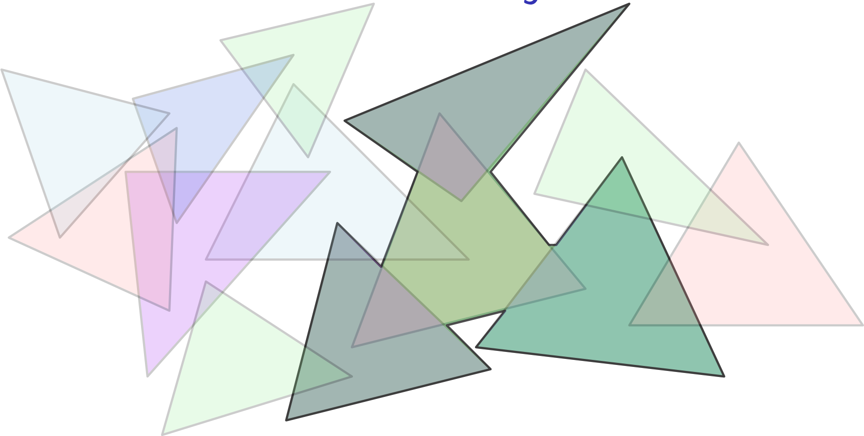
Minors of objects



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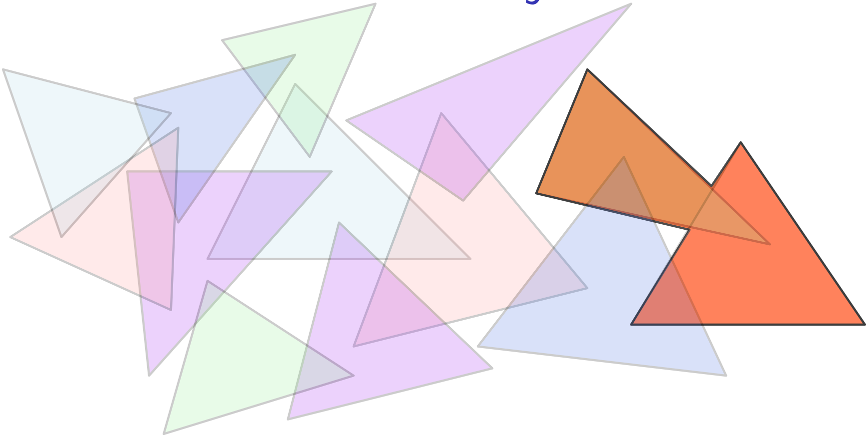

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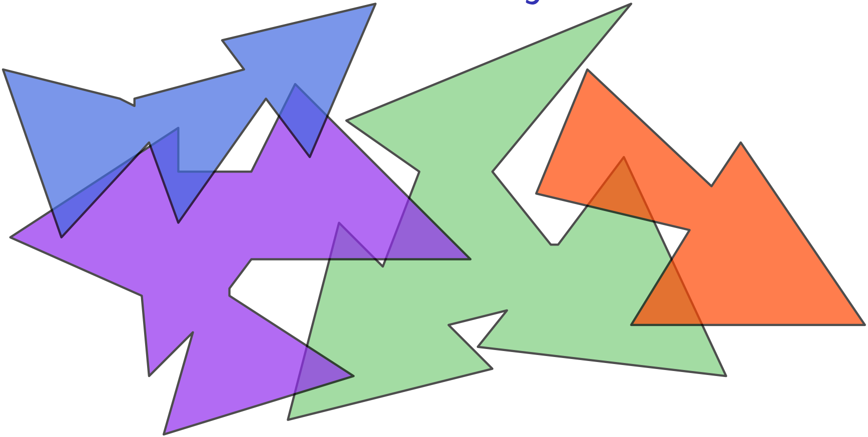
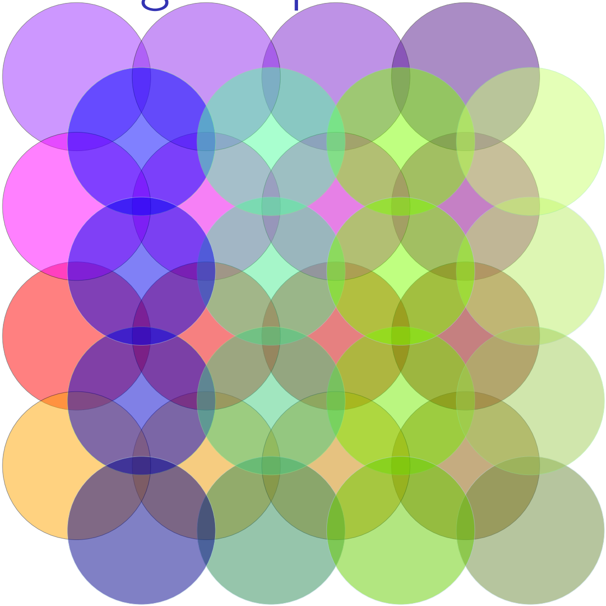
Minors of objects

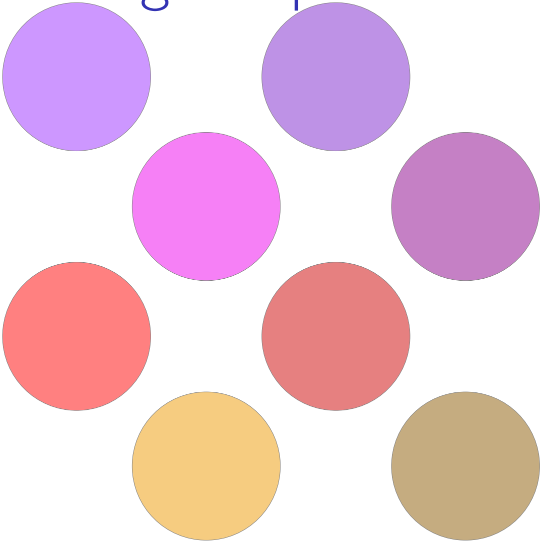


$\mathcal{G}$ is a **minor** of $\mathcal{F}$ if it can be obtained by deleting objects and taking unions of overlapping of objects.

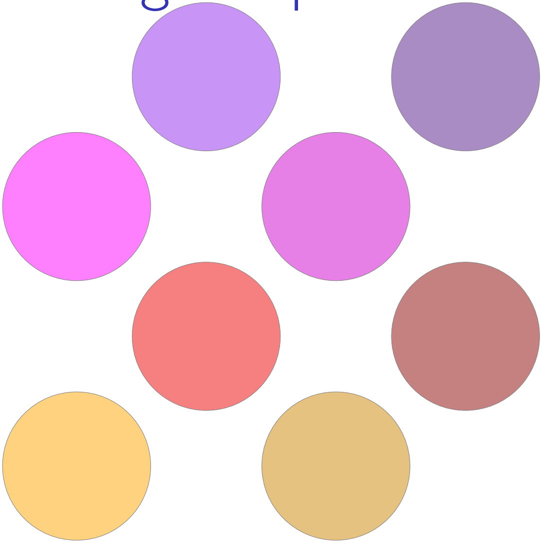
Large clique minors



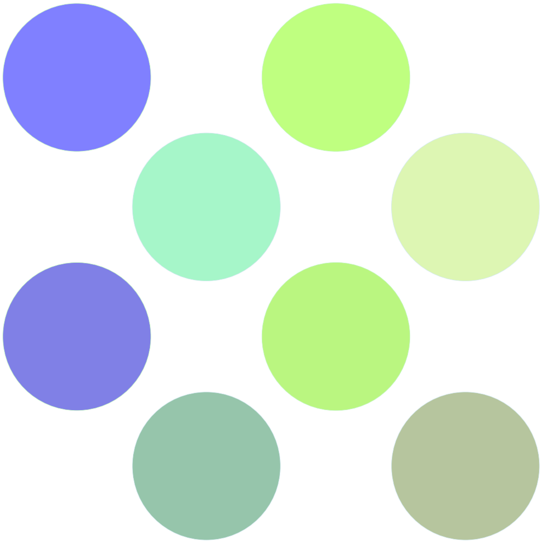
Large clique minors



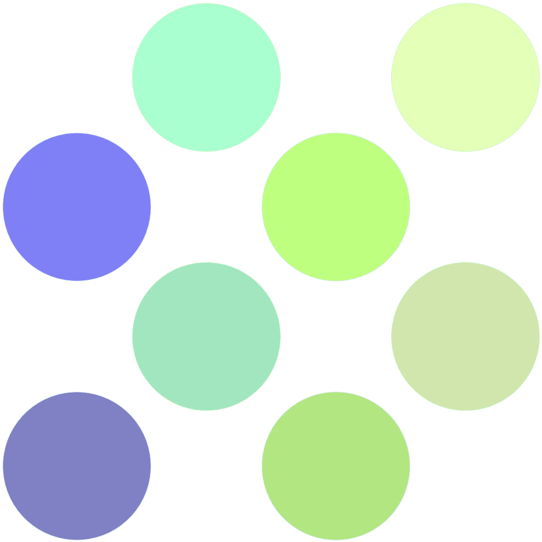
Large clique minors



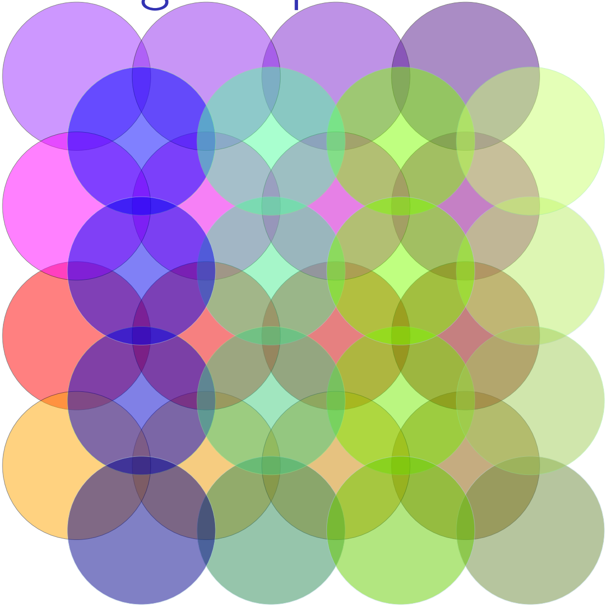
Large clique minors



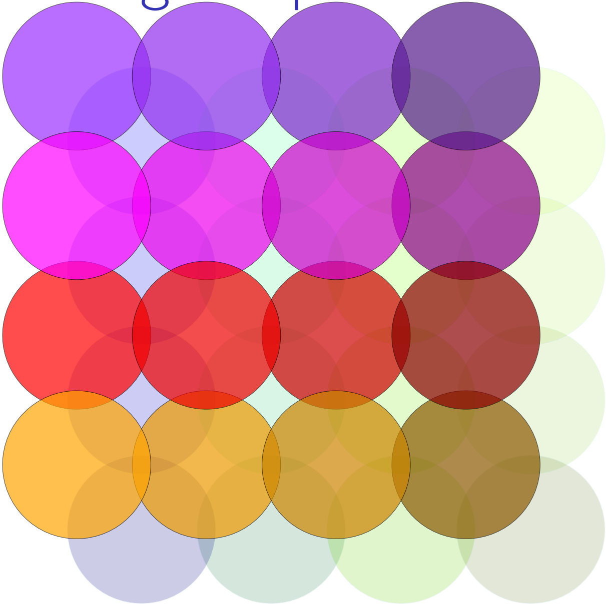
Large clique minors



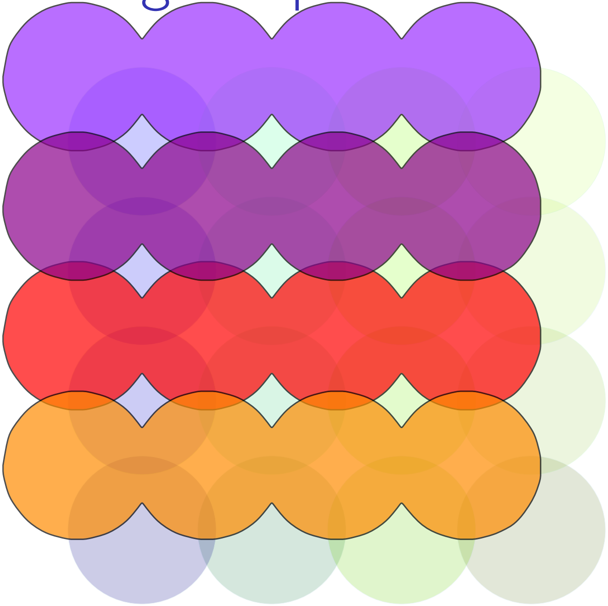
Large clique minors



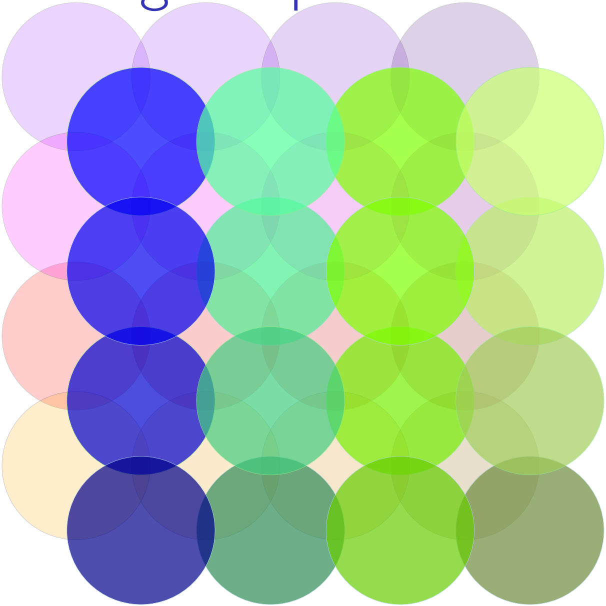
Large clique minors



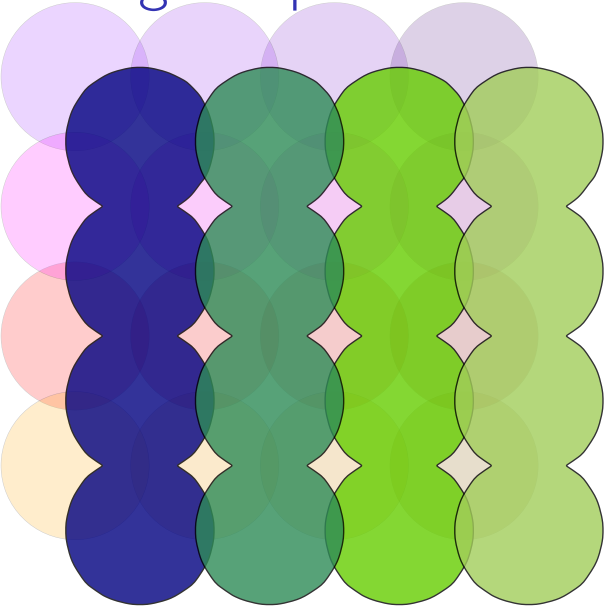
Large clique minors



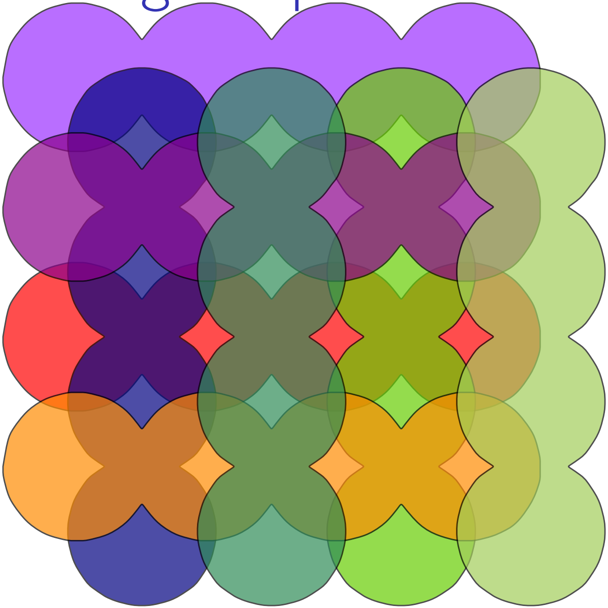
Large clique minors



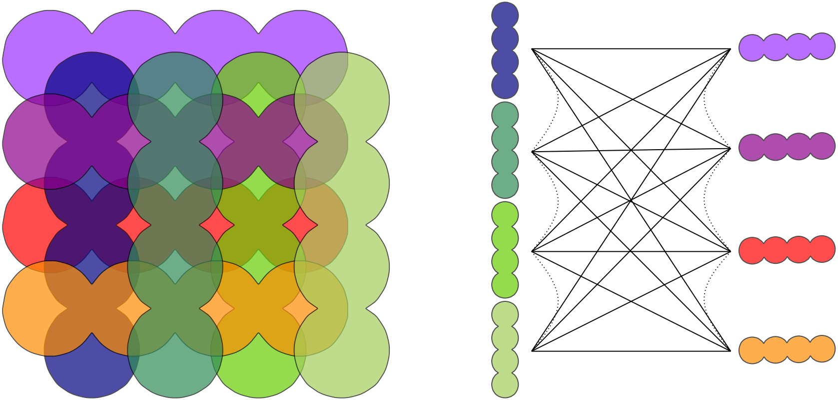
Large clique minors



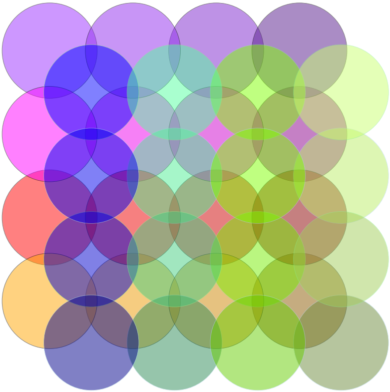
Large clique minors

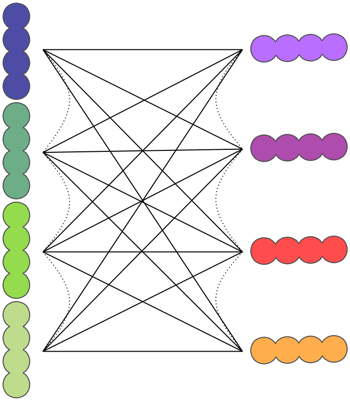


Large clique minors



Large clique minors





Main result: low-density

$\rho = O(1)$: PTAS for hitting set, set cover, subset dominating set

$\rho = \text{polylog}(n)$: QPTAS for same problems. No PTAS under ETH.

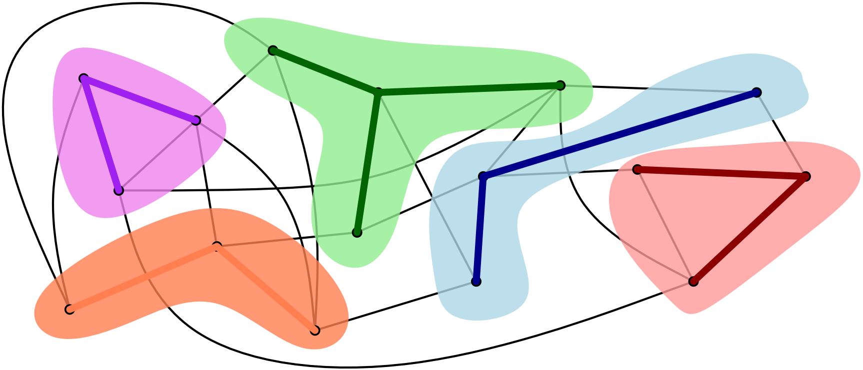
Main result: low-density

density ρ	$O(1)$	$\mathrm{polylog}(n)$	unbounded
hardness	NP-Hard	No PTAS	APX-Hard
algo	PTAS	QPTAS	

Main result: fat triangles

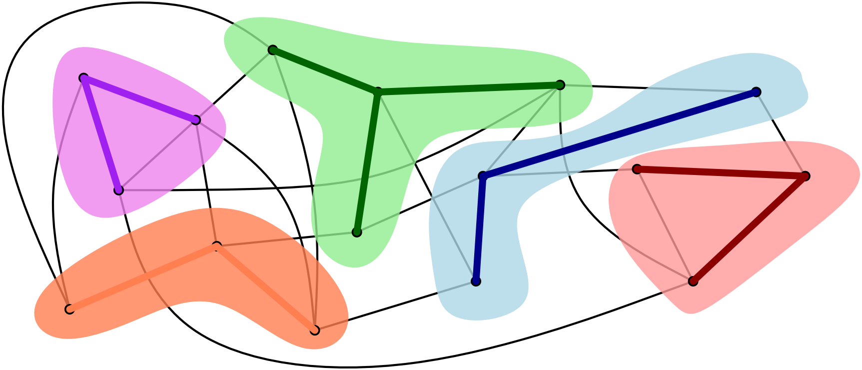
depth	$O(1)$	$\text{polylog}(n)$	unbounded
hardness	NP-Hard	No PTAS	APX-Hard
algo	PTAS	QPTAS	

Shallow minors



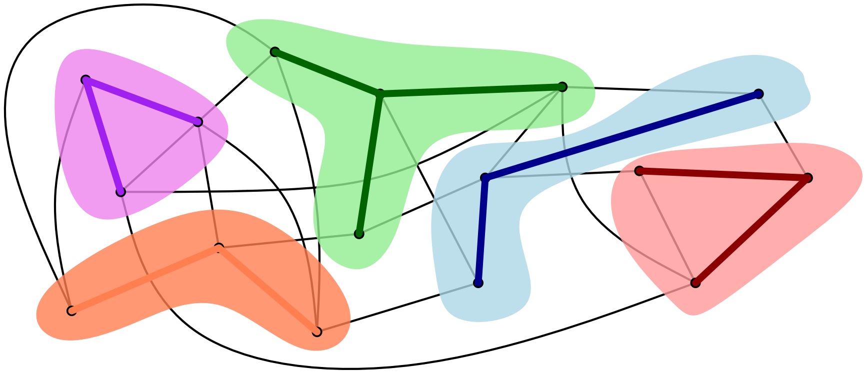
A vertex in H corresponds to a connected **cluster** of vertices in G.

Shallow minors



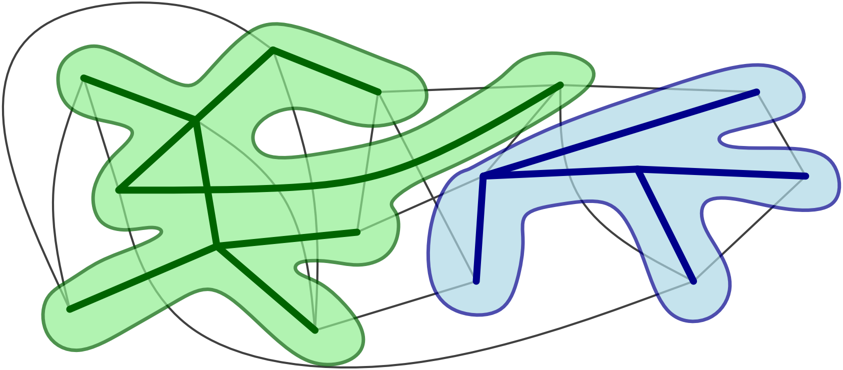
H is a **t-shallow minor** if each cluster induces a graph of radius t.

Shallow minors



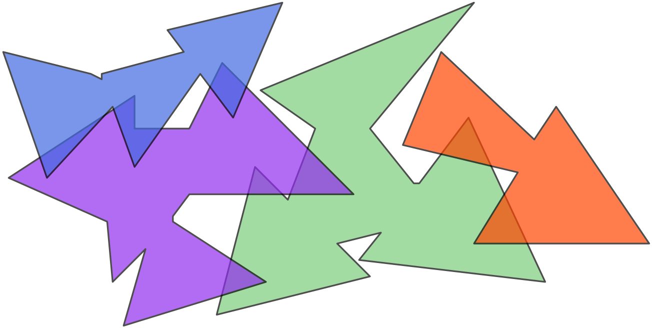
H is a **1-shallow minor** if each cluster induces a graph of radius 1.

Shallow minors



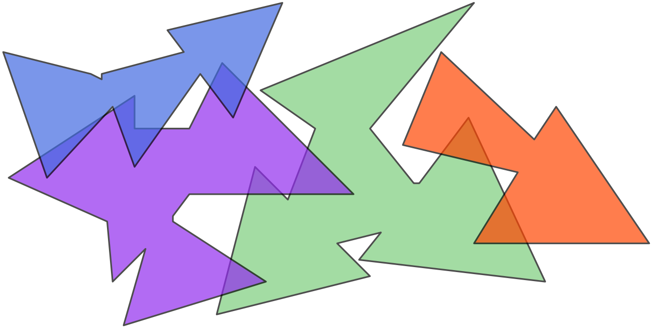
H is a **2-shallow minor** if each cluster induces a graph of radius 2.

Shallow minors of objects



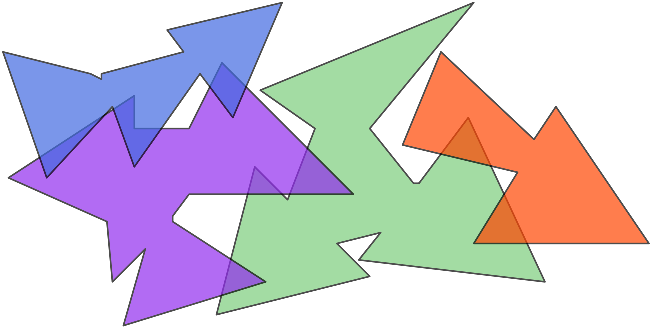
Each object in the minor corresponds to a **cluster** of objects in $\mathcal{F}$.

Shallow minors of objects



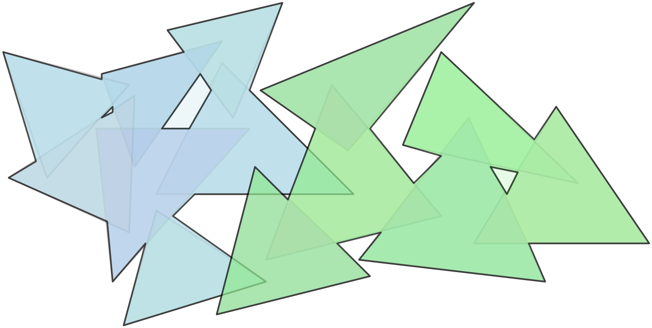
An object minor is a **t-shallow minor** if the intersection graph of each cluster has radius $\leq t$.

Shallow minors of objects



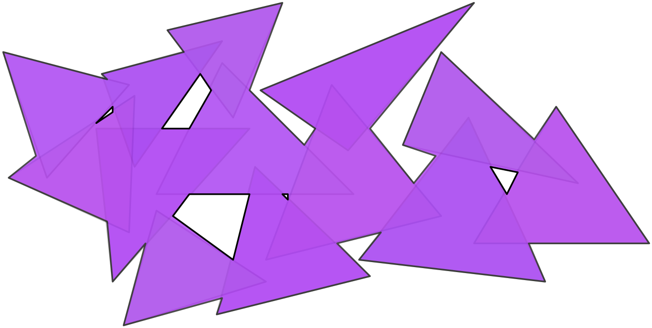
An object minor is a **1-shallow minor** if the intersection graph of each cluster has radius ≤ 1.

Shallow minors of objects



An object minor is a **2-shallow minor** if the intersection graph of each cluster has radius ≤ 2.

Shallow minors of objects



An object minor is a **3-shallow minor** if the intersection graph of each cluster has radius ≤ 3.

Shallow minors of low-density objects

A t-shallow minor of objects with density ρ has density $O(t^{O(d)}\rho)$

Shallow minors of low-density objects

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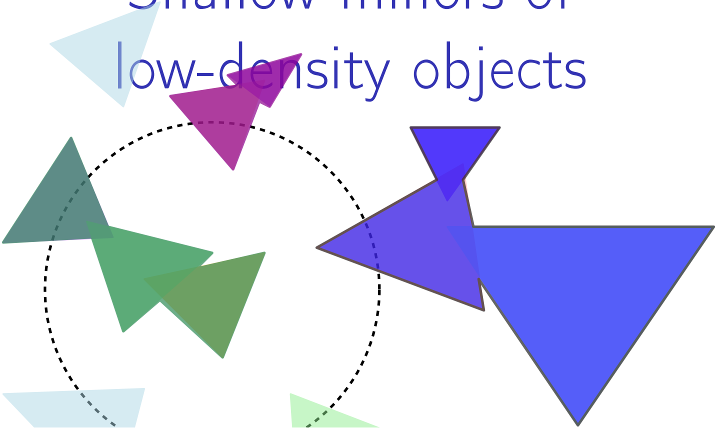
Shallow minors of low-density objects



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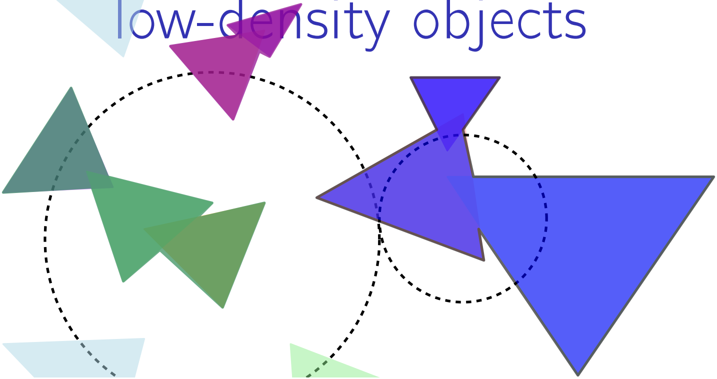
Shallow minors of low-density objects



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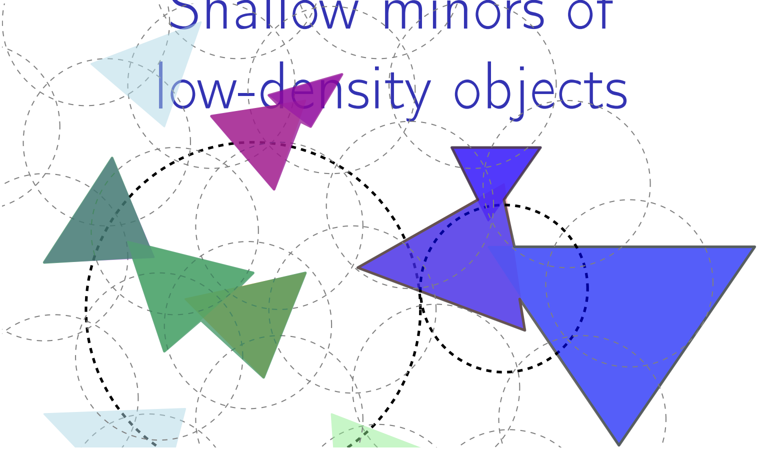
Shallow minors of low-density objects



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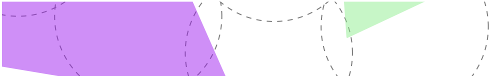
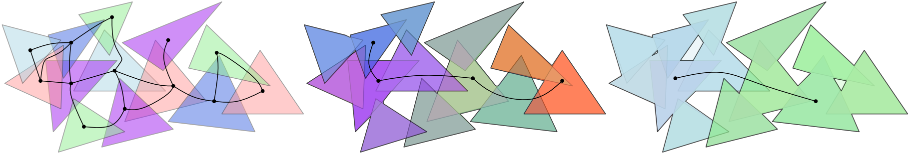


Shallow edge density

The r-**shallow density** of a graph is the max edge density over all r-shallow minors.

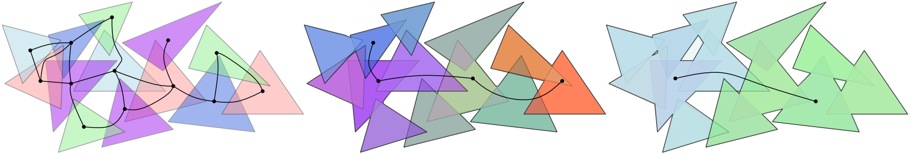


aka "greatest reduced average density"

[Nešetřil and Ossona de Mendez, 2008]

Expansion

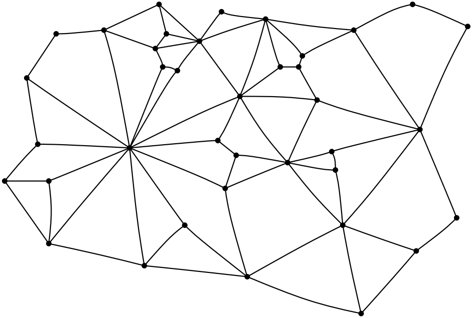
The **expansion** of a graph is the r-shallow density as a function of r.



Small expansion means shallow minors are sparse too.

[Nešetřil and Ossona de Mendez, 2008]

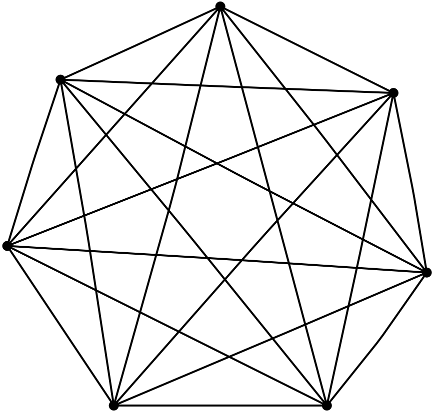
Examples of expansion



Planar graphs have constant expansion
(Euler's formula)

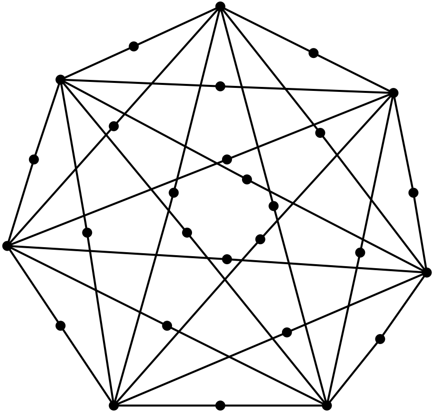
Minor-closed classes have constant expansion

Sparsity is not enough



Hide a clique by splitting the edges

Sparsity is not enough



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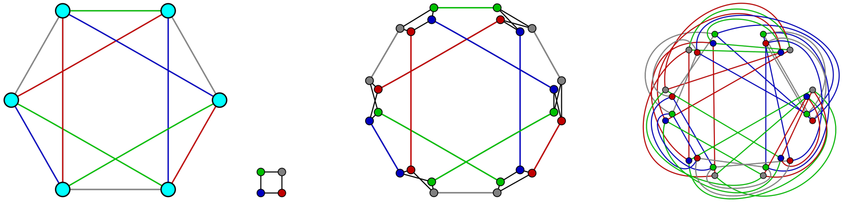

Constant degree expanders have high expansion...

Low density $\Rightarrow$ polynomial expansion

Graphs with density ρ have polynomial expansion $f(r) = O(\rho r^d)$

Low density ⇒ polynomial expansion

Graphs with density ρ have polynomial expansion $f(r) = O(\rho r^d)$

Main result: polynomial expansion

- Graph G with polynomial expansion
- *PTAS for (subset) dominating set*
- Extensions: multiple demands, reach, connected dominating set, vertex cover.

Conclusion

PTAS for low density graphs for dominating set type problems

Matching hardness w/r/t density

PTAS for polynomial expansion

thank you

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