Proximity in the Age of ~~Destruction~~ Distraction: Robust Approximate Nearest Neighbor Search

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2: k robust nearest neighbor

ANN = Approximate nearest neighbor

$\mathbf{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subseteq \mathbb{R}^{\boldsymbol{d}}$: Points in high dimensions.
$\boldsymbol{k}$: Parameter
Q: Given query $\boldsymbol{q}$, find ANN in $\mathbf{P}$, when ignoring $\boldsymbol{k}$ largest coordinates.

3: Robust distance

But without the noisy coordinates

Definition

$\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$
$\pi = \mathrm{sort}(\boldsymbol{x})$: $|x_{\pi(1)}| \geq |x_{\pi(2)}| \geq \cdots \geq |x_{\pi(d)}|$.
i-tail of $\boldsymbol{x}$: $\boldsymbol{x}_{\setminus i} = \big(0, \ldots, 0, |x_{\pi(i+1)}|, |x_{\pi(i+2)}|, \ldots, |x_{\pi(d)}|\big)$.

computing $\boldsymbol{x}_{\setminus i}$ takes $O(d)$ time.

$x, y \in \mathsf{P}$: $\mathrm{dist}_{\setminus k}(x, y) = \|(x - y)_{\setminus k}\|$

Definition

k-robust nearest-neighbor to $q \in \mathbb{R}^d$ in P:

$$\mathrm{nn}_{\setminus k}(\mathsf{P}, q) = \arg\min_{y \in \mathsf{P}} \mathrm{dist}_{\setminus k}(q, y)$$

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1. High dim data common.
2. Some coordinates of $\boldsymbol{x}$ and $\boldsymbol{v}$ are irrelevant features.
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5: Results I

Reduction from Robust ANN to ANN.

1. General reduction from the robust ANN problem to the "standard" ANN problem.
2. L_1: $\exists q^* \in \mathbf{P}$ s.t. $\mathbf{dist}_{\setminus k}(q, q^*) \leq r \implies$
   1. $\delta \in (0,1)$: parameter
   2. return x s.t. $\mathbf{dist}_{\setminus O(k/\delta)}(q, x) \leq O(r/\delta)$
   3. $O(n^\delta)$ ANN queries in 2-ANN DS.
3. L_ρ: $\exists$NN with dist $\leq r$
   1. report point in distance $O(r(c + \frac{1}{\delta})^{1/\rho})$
   2. ignoring $O(k(\frac{1}{\delta} + c))$ coordinates
   3. n^δ of $c^{1/\rho}$-ANN queries.

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$(1+\varepsilon)$-approximation

L_1: $\exists q^* \in \mathbf{P}$ s.t. $\mathbf{dist}_{\parallel k}(q, q^*) \leq r \implies$

1. $\varepsilon, \delta \in (0,1)$: parameters
2. return x s.t. $\mathbf{dist}_{\parallel O(\frac{k}{\delta\varepsilon})}(q, x) \leq (1+\varepsilon)r$
3. $\widetilde{O}(n^{\delta}/\varepsilon)$ ANN queries in $(1+O(\varepsilon))$-ANN DS.

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7: Results III

Budgeted version

1. Every coordinate has a cost of ignoring it.
2. $\exists \boldsymbol{q}^* \in \mathbf{P}$, s.t., $\mathbf{dist}_{\|_1}(\boldsymbol{q}, \boldsymbol{q}^*) \leq r \implies$
3. Algorithm returns a point in distance $O(r)$
4. Cost of ignored coordinates is $O(1)$.
5. in L_1 norm.
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8: Results III

Data sensitive LSH queries

1. known ANN DS perform better in practice.
   1. Many cases return *exact* NN!
   2. Find it early!
   3. Spend more time "proving" its ANN.
2. **Q:** If query is easy, why work hard?
3. **Result:** (for $\{0,1\}^d$).
   1. query time: $O(d \exp(\Delta) \log n)$
   2. Δ is smallest value s.t.

$$\sum_{i=1}^{n} \exp\left(-\Delta \frac{\mathrm{dist}(q, x_i)}{r}\right) \leq 1,$$

   3. works quickly on low dimensional data.

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Part I

Locality sensitive hashing revisited

9: Locality sensitive hashing (LSH)

1. Technique due to Indyk and Motwani **[Indyk and Motwani, 1998]**.
2. Input: $\mathbf{P} \subseteq \{0,1\}^d$: set of n points.
3. Parameters r and ε.
4. **Q:** Given query point $q \in \{0,1\}^d$:
   1. If $\mathbf{d}_H(q, \mathbf{P}) \leq r$ then return $v \in \mathbf{P}$ s.t. $\mathbf{d}_H(q, v) \leq (1+\varepsilon)r$.
   2. If $\mathbf{d}_H(q, \mathbf{P}) > (1+\varepsilon)r$ then return "$\mathbf{d}_H(q, \mathbf{P}) > r$".
   3. If $\mathbf{d}_H(q, \mathbf{P}) \in \big(r, (1+\varepsilon)r\big)$ return either of the above.

10: A random projection...

1. Sequence $m \equiv i_1, i_2, \ldots, i_\ell \in [\![d]\!] = \{1, \ldots, d\}$.
2. $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$
3. **projection** by m is

$$mx = (x_{i_1}, \ldots, x_{i_\ell}) \in \mathbb{R}^\ell.$$

4. $mP = \{mx \mid x \in P\}$.

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11: A splay

1. τ: a probability
2. include ith coordinate, for $i = 1, \ldots, d$, in m with probability τ
3. $\mathcal{D}_\tau$: Distribution of such projections.
4. $m = i_1, \ldots, i_\ell$ and $n = j_1, \ldots, j_{\ell'}$:
   $m|n = i_1, \ldots, i_\ell, j_1, \ldots, j_{\ell'}$.
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12: Preprocessing

1. Set $\beta = 1/(1+\varepsilon) \in (0,1)$.
2. $\tau = 1/r, \quad t = \beta \log n, \quad$ and $L = O(n^{\beta} \log n)$.
3. Pick randomly L sequences $M_1, \ldots, M_L \in \mathcal{D}_{\tau}^{t}$.
4. Algorithm computes $\mathbf{Q}_i = M_i \mathbf{P}_i$, for $i = 1, \ldots, L$.
5. Store $\mathbf{Q}_i$ in a hash table $\mathcal{H}_i$ for $i = 1, \ldots, L$.
6. **Answering a query.**
   1. Given $q \in \{0,1\}^d$: compute $q_i = M_i q$, for $i = 1, \ldots, L$.
   2. Retrieve list ℓ_i from $\mathcal{H}_i$, of all points that collide with q_i.
   3. Scan $\ell_1, \ldots, \ell_L$.
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13: What is going on?

Collision probabilities

$\boldsymbol{q}$, $\boldsymbol{y}$: Two points.

$\varepsilon = 1 \qquad x = \mathbf{d}_H(\boldsymbol{q}, \boldsymbol{y}). \qquad t = \frac{\log n}{1+\varepsilon}$

$$f(x) = \mathbf{Pr}[M\boldsymbol{q} = M\boldsymbol{x}] = \left(\left(1 - \frac{1}{r}\right)^x\right)^t \approx \mathbf{exp}\left(-\frac{xt}{r}\right)$$

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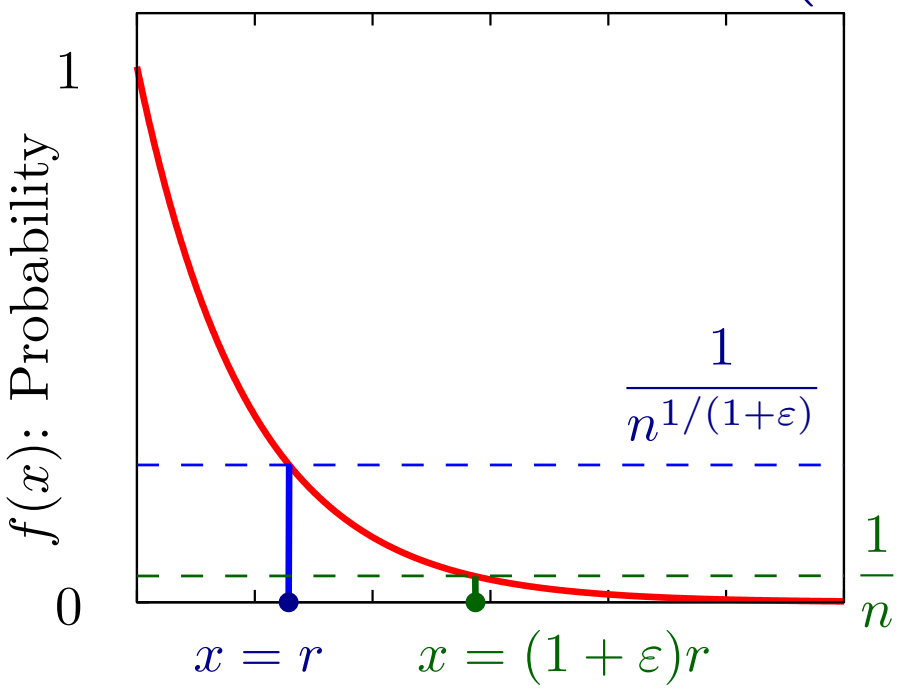

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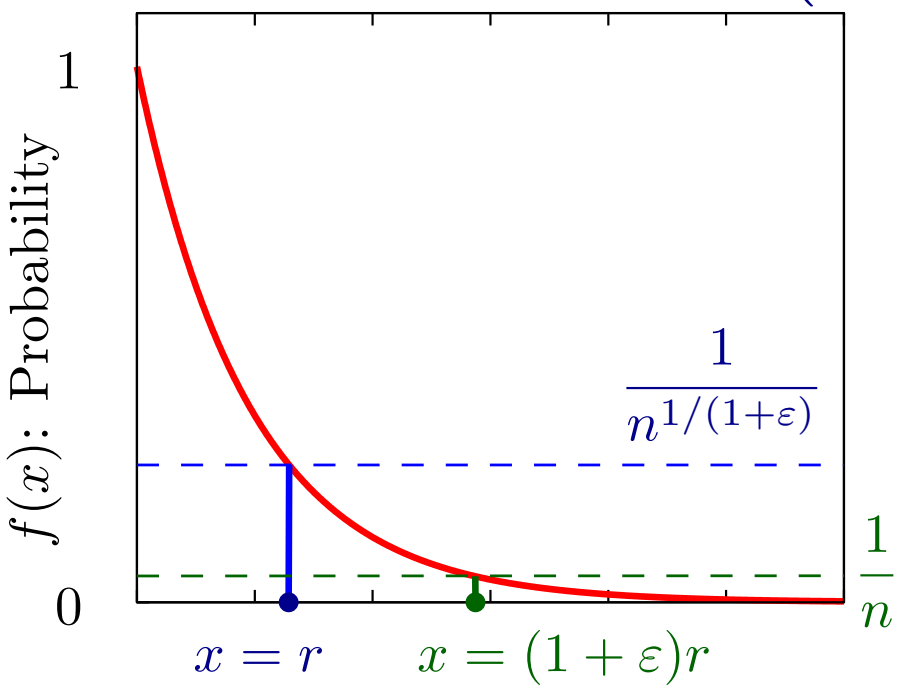

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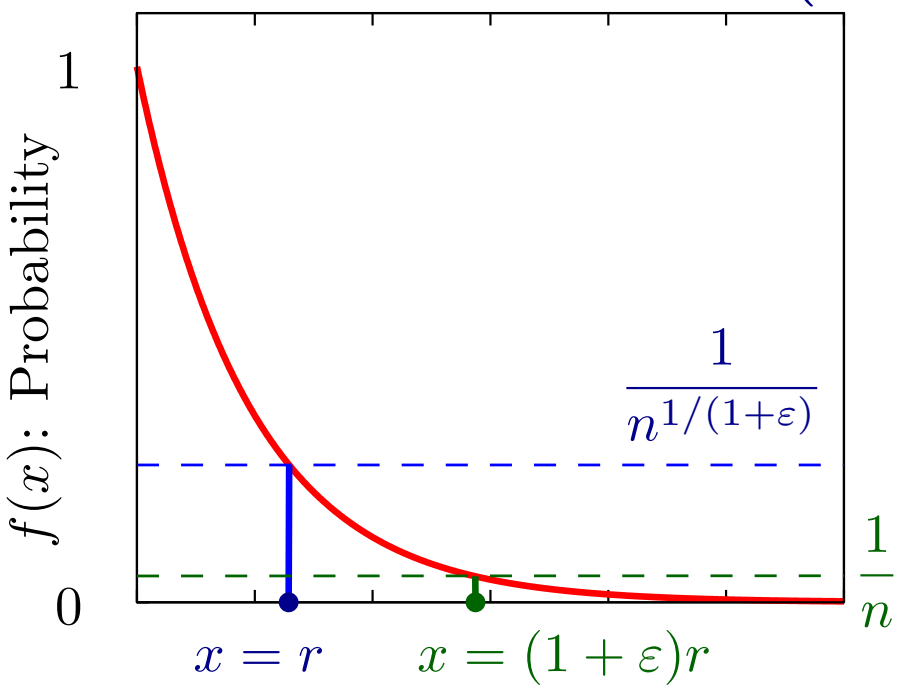


1. $f(x) = \exp\left(-\frac{x \log n}{(1+\varepsilon)r}\right)$
2. $f(r) = 1/n^{1/(1+\varepsilon)}$

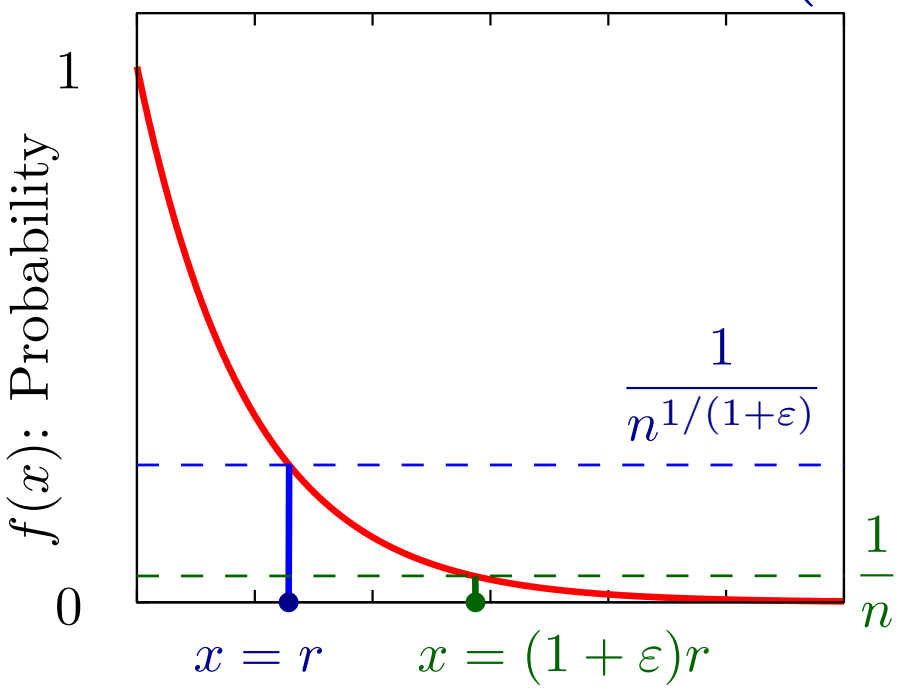
13: What is going on?

Collision probabilities

$\boldsymbol{q}$, $\boldsymbol{y}$: Two points.

$\varepsilon = 1$ $\quad x = \mathbf{d}_H(\boldsymbol{q}, \boldsymbol{y})$. $\quad t = \frac{\log n}{1+\varepsilon}$

$$f(x) = \mathbf{Pr}[M\boldsymbol{q} = M\boldsymbol{x}] = \left(\left(1 - \frac{1}{r}\right)^x\right)^t \approx \exp\left(-\frac{xt}{r}\right)$$



1. $f(x) = \exp\left(-\frac{x \log n}{(1+\varepsilon)r}\right)$
2. $f(r) = 1/n^{1/(1+\varepsilon)}$
3. $f\Big((1+\varepsilon)r\Big) = 1/n$

14: Correctness – avoiding few bad coordinates

Lemma

K: set of r forbidden coordinates. Probability $M = (m_1, \ldots, m_t) \in \mathcal{D}_\tau^t$ does not sample any coordinate of K is $\geq n^{-\beta(1+1/(2r))} \approx 1/n^\beta = 1/n^{1/(1+\varepsilon)}$.

Proof.

Probability m_i not contain any of coordinates of K is $(1-\tau)^r$.
Desired probability is

$$\left(1-\frac{1}{r}\right)^{rt} = \left(1-\frac{1}{r}\right)^{r\beta \log n} \geq \exp(-\beta \log n)\left(1-\frac{1}{r}\right)^{\beta \log n}$$
$$\geq n^{-\beta} \exp\left(-\frac{1}{2r}\beta \log n\right) = n^{-\beta-\beta/(2r)},$$

by $(1-1/m)^{m-1} \geq \frac{1}{e} \geq (1-1/m)^m$, $e^{-2x} \leq 1-x \leq e^x$. □

15: Proof

Lemma

(A) *If* $\mathbf{d}_H(q, \mathbf{P}) \leq r$. *W.h.p. return* $x \in \mathbf{P}$ *s.t.* $\mathbf{d}_H(q, x) \leq (1+\varepsilon)r$.

(B) *If* $\mathbf{d}_H(q, \mathbf{P}) \geq (1+\varepsilon)r$. *Then,* $\mathbf{E}[\sum_i |\ell_i|] = O(L)$.

Proof.

(A) By previous lemma. Probability $M_i x = M_i q$ is at least $1/n^\beta$. Probability all L data-structures fails, is $\leq (1 - 1/n^\beta)^L < 1/n^{O(1)}$, as $L = O(n^\beta \log n)$.

(B) For $v \in \mathbf{P}$, s.t. $\mathbf{d}_H(v, q) \geq (1+\varepsilon)r$. Probability for $M \in \mathcal{D}_\tau^t$ misses all the $\geq (1+\varepsilon)r$ coordinates (i.e., collision!) is

$$\leq (1-\tau)^{(1+\varepsilon)rt} \leq \exp\left(-(1+\varepsilon)\frac{tr}{r}\right) =$$

$\exp(-(1+\varepsilon)\beta \log n) = 1/n$, as $\beta = 1/(1+\varepsilon)$. Expected size of ℓ_i is $\leq n \times (1/n) = 1$. □

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16: LSH: Query running time

1. Compute $\forall i \quad M_i q$. Takes $O(dt) = O(d \log n)$ time.
2. Repeat above L times: $O(Ld \log n) = O\left(n^{1/(1+\varepsilon)} \log^2 n\right)$ time overall.
3. In expectation, after scanning $O(L)$ points, either encountered point in distance r, or all points are in distance larger than $(1+\varepsilon)r$.
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Part II

Robust nearest neighbor

17: The good, the bad, and the truncated.

1. technique probes coordinates, trying to detect "hidden" mass of a bad point.
2. mass might concentrate in few coordinates (i.e., $k+1$).
3. Such point still good?
4. Point one has to ignore many more than k coordinates is "bad".
5. Bound the influence of a single coordinate on distance to q.
6. $r = \mathrm{dist}_{\setminus k}(q, \mathsf{P})$.
7. Truncate coordinate of $x - q$ if larger than r/k.

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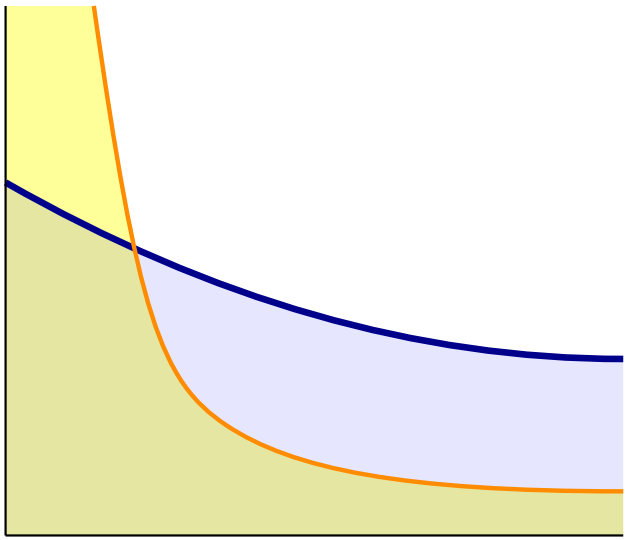

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2. Prove good approximation (slightly refined LSH analysis).
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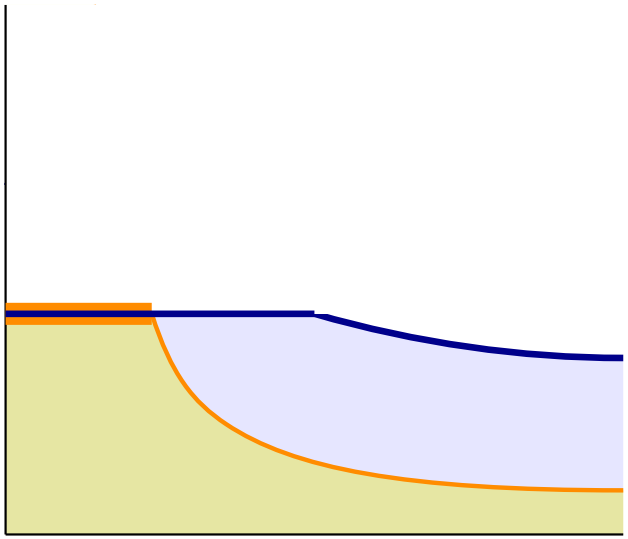

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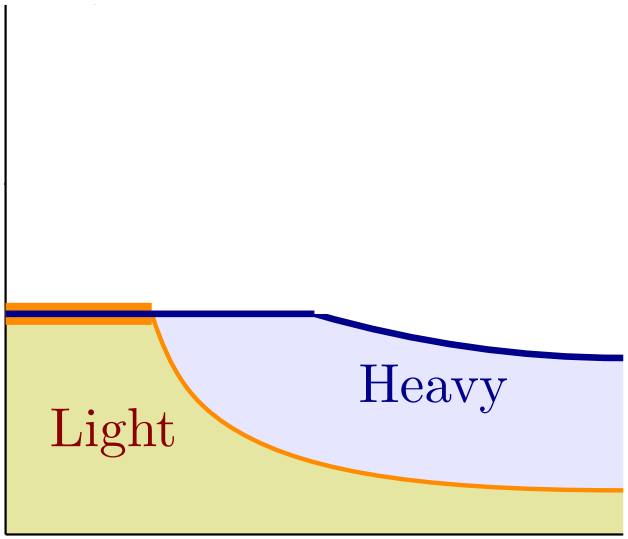

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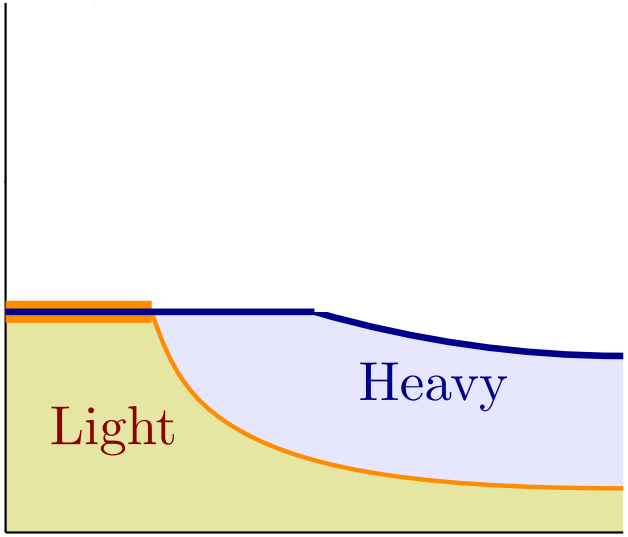

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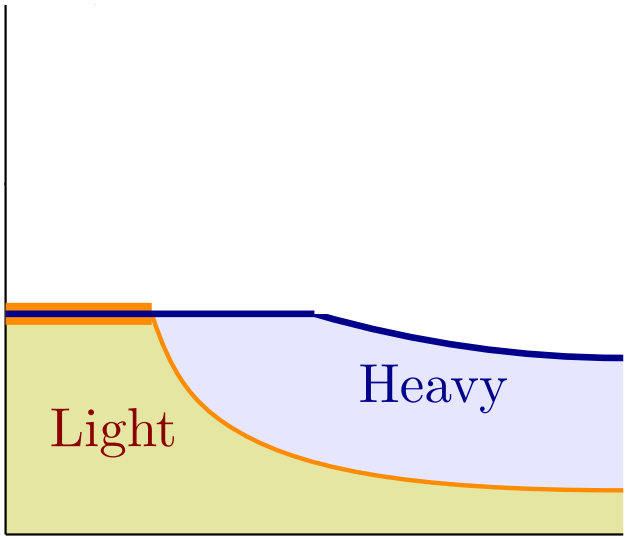

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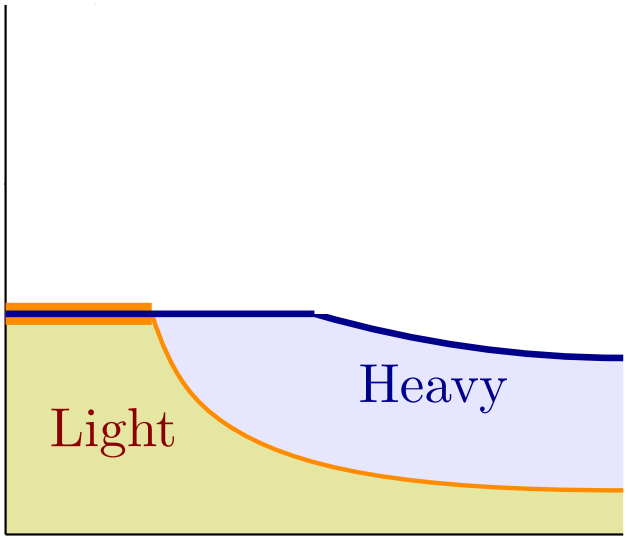

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   Returns a point where ignore slightly more coordinates than k
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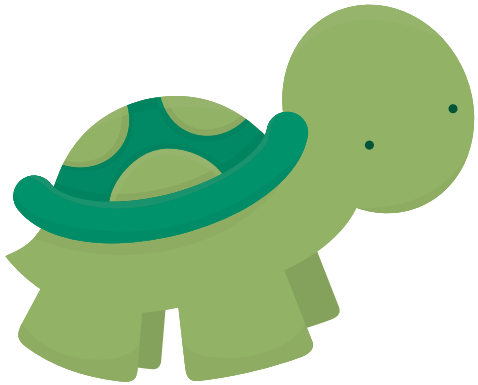
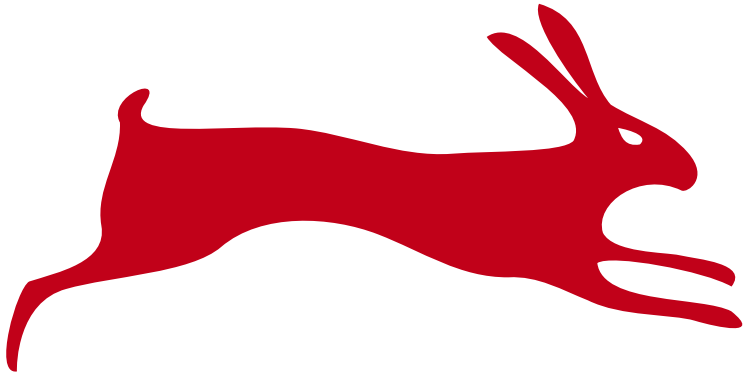
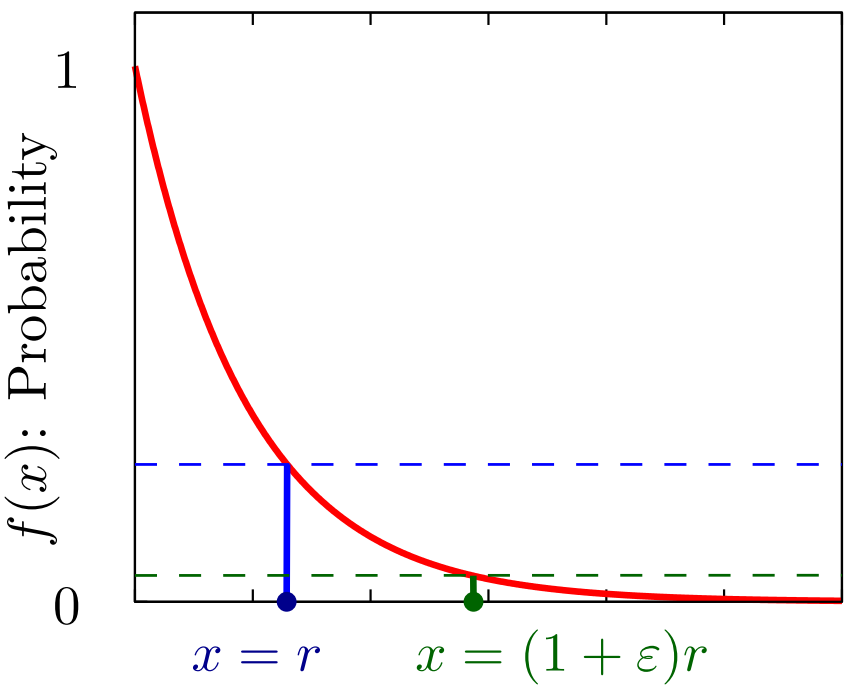
Part III

Data sensitive LSH

20: The tortoise and the hare

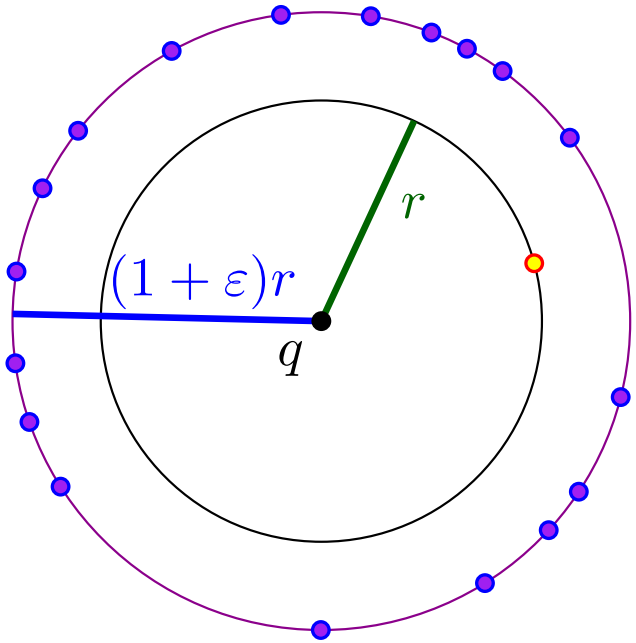






1. Pick coordinate prob $1/d$
2. Repeat t times to "push away" far points. $f((1+\varepsilon)r) \leq 1/n$.
3. Repeat $L \approx 1/f(r)$ times, to succeed w.h.p.

21: The worst/good case for LSH



Worst case

Work hard to push away a single point...

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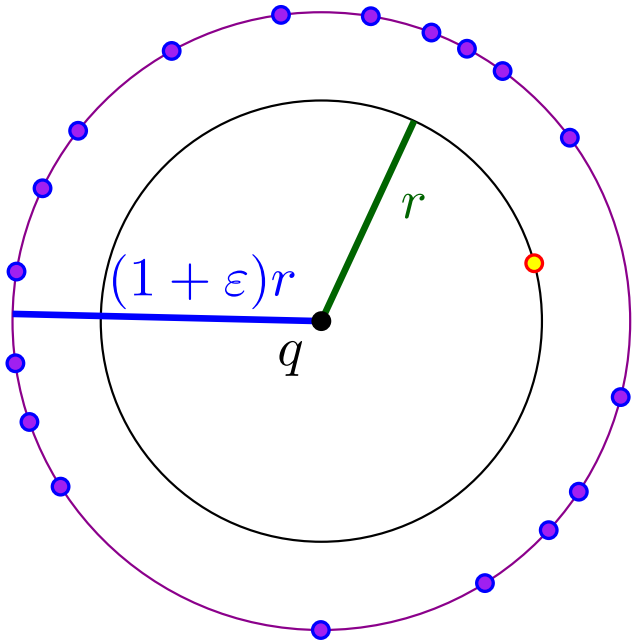

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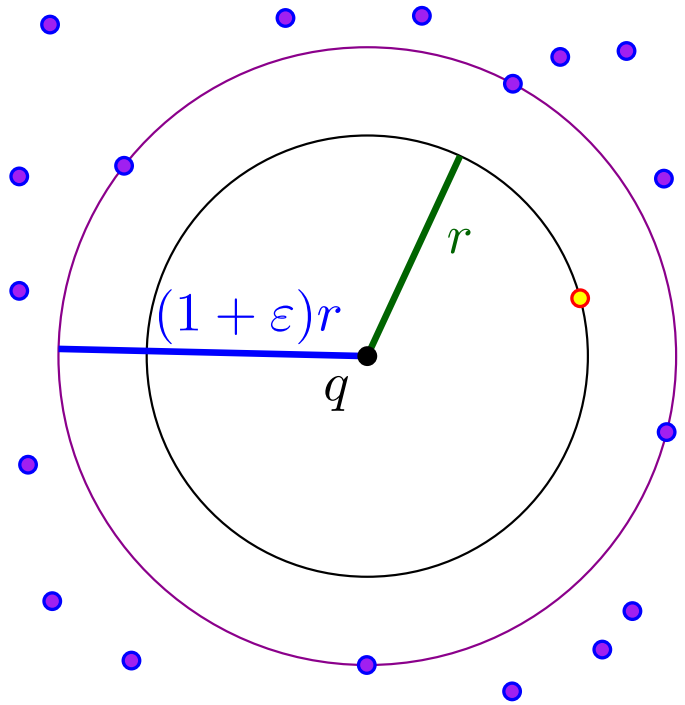

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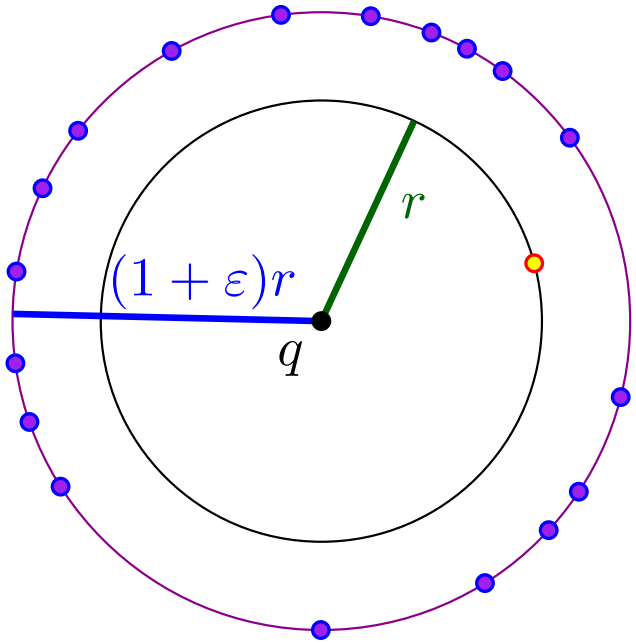

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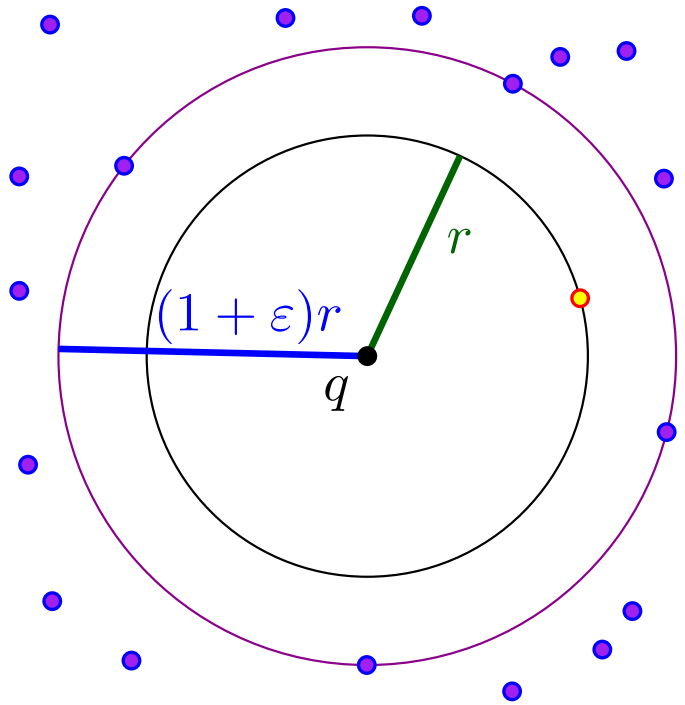

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22: Data sensitive LSH.

1. **Idea**: interpret LSH as estimator of local density.
2. If data set is sparse near query point, LSH would hit NN point quickly...
3. ...density estimation would tell us that there are no more close by points.
4. Do exponential search – start with an insensitive LSH scheme (that is fast).
5. ... use more sensitive LSHs, till density estimation tell us we are done.
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23: Setup

1. $\mathbf{P} \subseteq \{0,1\}^d$: n points.
   r: radius, $\varepsilon > 0$.
2. Preprocessing:
   $i = 1, \ldots, \mathcal{N} = \frac{\log n}{1+\varepsilon}$:
   1. $\mathrm{f}_i(\ell) = (1 - 1/r)^{\ell i} \leq \exp(-\ell i / r)$
   2. $\mathrm{s}(i) = \frac{c \log n}{\mathrm{f}_i(r)} = O\left(e^i \log n\right)$,
   3. $\mathcal{F}_i = \left\{ m_1, m_2, \ldots, m_{\mathrm{s}(i)} \in \mathcal{D}^i_{1/r} \right\}$
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2. For $i = 1 \ldots$
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    2. If $X_i > 4s(i)$ then $i++$, continue to next iteration.
    3. If $X_i \leq 4s(i)$, then
        1. extract the X_i points
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Expected number of collisions with $\boldsymbol{q}$, for a $m \in \mathcal{F}_i$ is

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Expected number collisions with $\boldsymbol{q}$ (with all projections $\mathcal{F}_i$):
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If scan collision lists in ith level: query time $O\big(d(X_i + \mathbf{s}(i))\big)$.

If $X_i = O(\mathbf{s}(i))$, then query time is small.

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Γ_i depends on how the data looks like near the query.

Assume locally near query point, data is a uniform low dimensional point set.

Assume # points in distance ℓ from $q \leq \#(\ell) = O\big((\ell/r)^k\big)$.
k: small, r: distance to NN to q.

$\Gamma_i = \mathrm{s}(i) \sum_{x \in P} \mathrm{f}_i\big(\mathrm{d}_H(q,x)\big)$
$\leq O(e^i \log n) \sum_{j=1}^{\infty} \#(jr) \exp(-ji)$
$= \sum_{j=1}^{\infty} O\big(j^k \exp(i - ji) \log n\big).$

Setting $i = k$:

$$\Gamma_k \leq \sum_{j=1}^{\infty} O\left(\left(\frac{j}{e^{j-1}}\right)^k \log n\right) \leq \sum_{=1}^{\infty} O\left(\frac{j}{2^j} \log n\right) \leq O(\log n)$$

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Lemma

If $\exists$ point with distance $\leq r$ from q, then algorithm computes, w.h.p., point x s.t., $\mathbf{d}_H(q, x) \leq (1+\varepsilon)r$.

Proof.

$q^* \in P$: nearest neighbor to q.

For DS_i: probability q^* not collide with q
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$i < \mathcal{N}$: Algorithm stops and scans all lists.

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$\leq (1-f_i(r))^{s(i)} \leq \exp(-f_i(r)s(i)) = \exp(-O(\log n)) \leq 1/n^{O(1)}$.

Since $s(i) = O(e^i \log n)$

$i < \mathcal{N}$: Algorithm stops and scans all lists.

$i = \mathcal{N}$: Just reg LSH. **(Approximation only in this case!)**

27: Analysis

Lemma

If $\exists$ point with distance $\leq r$ from q, then algorithm computes, w.h.p., point x s.t., $d_H(q,x) \leq (1+\varepsilon)r$.

Proof.

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28: Value of $\mathbf{s(i)}$

Lemma

For any $i > 0$, *we have* $\mathbf{s}(i) = O\left(e^i \log n\right)$.

Proof.

$$\mathbf{f}_i(r) = (1 - 1/r)^{(r-1)ri/(r-1)} \geq \exp\left(-r\tfrac{i}{r-1}\right) = \exp\left(-i - \tfrac{i}{r-1}\right)$$

since $\forall m > 0$: $(1 - 1/m)^{m-1} \geq 1/e \geq (1 - 1/m)^m$.

Can assume that $r > \log n \geq i$ (duplicate coordinates)...

$\implies \mathbf{f}_i(r) = \Theta(\exp(-i))$.

$\implies \mathbf{s}(i) = O(\frac{\log n}{\mathbf{f}_i(r)}) = O\left(e^i \log n\right)$. □

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29: Worst case query time

Lemma

For a query point $\boldsymbol{q}$*, the worst case query time is* $O(dn^{1/(1+\epsilon)} \log n)$*, with high probability.*

30: Data-dependent query time

Lemma

$\mathbf{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subseteq \mathbb{H}^d$
$\ell_i = \mathbf{d}_H(\boldsymbol{q}, \boldsymbol{x}_i)$, *for* $\boldsymbol{i} = \mathbf{1}, \ldots, \boldsymbol{n}$,
$\boldsymbol{\Delta}$*: smallest value s.t.*

$$\sum_{i=1}^{n} \mathbf{exp}\left(-\frac{\boldsymbol{\Delta}\ell_i}{r}\right) \leq \mathbf{1}.$$

$\Longrightarrow$ *expected query time* $O\left(de^{\boldsymbol{\Delta}} \log n\right)$.

31: Proof

Condition implies: $\gamma_j \leq \gamma_\Delta \leq 1$, for any $j \geq \Delta$.

$\Longrightarrow$ $j \geq \Delta$: $\Gamma_j = \mathbf{E}[X_j] = \gamma_j s(j) \leq s(j)$.

Markov's inequality, $\mathbf{Pr}[X_j \leq 4s(j)] \geq 1 - 1/4$.
Then the algorithm would terminate in this iteration.

$Y_j = 1$: indicator the algorithm reached jth iteration.

Algorithm files in iterations $\Delta, \Delta + 1, \ldots, j - 1$.

$\Longrightarrow$ $\mathbf{Pr}[Y_j = 1] \leq 1/4^{j-\Delta}$.

jth iteration (if happens): $O(ds(j))$ time.

Overall expected running time is $O\left(\mathbf{E}\left[d \sum_{j=1}^{\mathcal{N}} Y_j s(j)\right]\right)$.

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$$O\left(d\sum_{j=1}^{\Delta} s(j) + d\sum_{j=\Delta+1}^{\mathcal{N}} \frac{s(j)}{4^{j-\Delta}}\right)$$

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33: The result

$\mathbf{P} \subseteq \mathbb{H}^d = \{0,1\}^d$: set of n points,
$\varepsilon > 0$ and r.
Preprocess in $O\left(dn^{1+1/(1+\varepsilon)} \log n\right)$ time/ space.

Given $q \in \mathbb{H}^d$, decide (approximately) if $\mathrm{d}_H(q, \mathbf{P}) \leq r$, in $O\left(dn^{1/(1+\varepsilon)} \log n\right)$.

If query is "easy", DS returns *exact* nearest neighbor.

If $\mathrm{d}_H(q, \mathbf{P}) \leq r$, and $\exists \Delta < (\log n)/(1+\varepsilon)$ s.t. $\sum_{x \in \mathbf{P}} \exp(-\mathrm{d}_H(q,x)\Delta/r) \leq 1$

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34: Remarks

1. If data is k dimensional (bounded growth)
   Then query time $O(d \log n)$.
   Known: **[Datar, Immorlica, Indyk, and Mirrokni, 2004]**.
   New DS more general. Handles this case with no modification.
2. Fine tuning the LSH scheme to the hardness of the given data is not a new idea.
   1. **[Andoni, Datar, Immorlica, Indyk, and Mirrokni, 2006]**
   2. Fine tune LSH construction parameters for set of queries, to optimize overall query time.
   3. New DS: adapts parameters on the fly during query process, depending on query.

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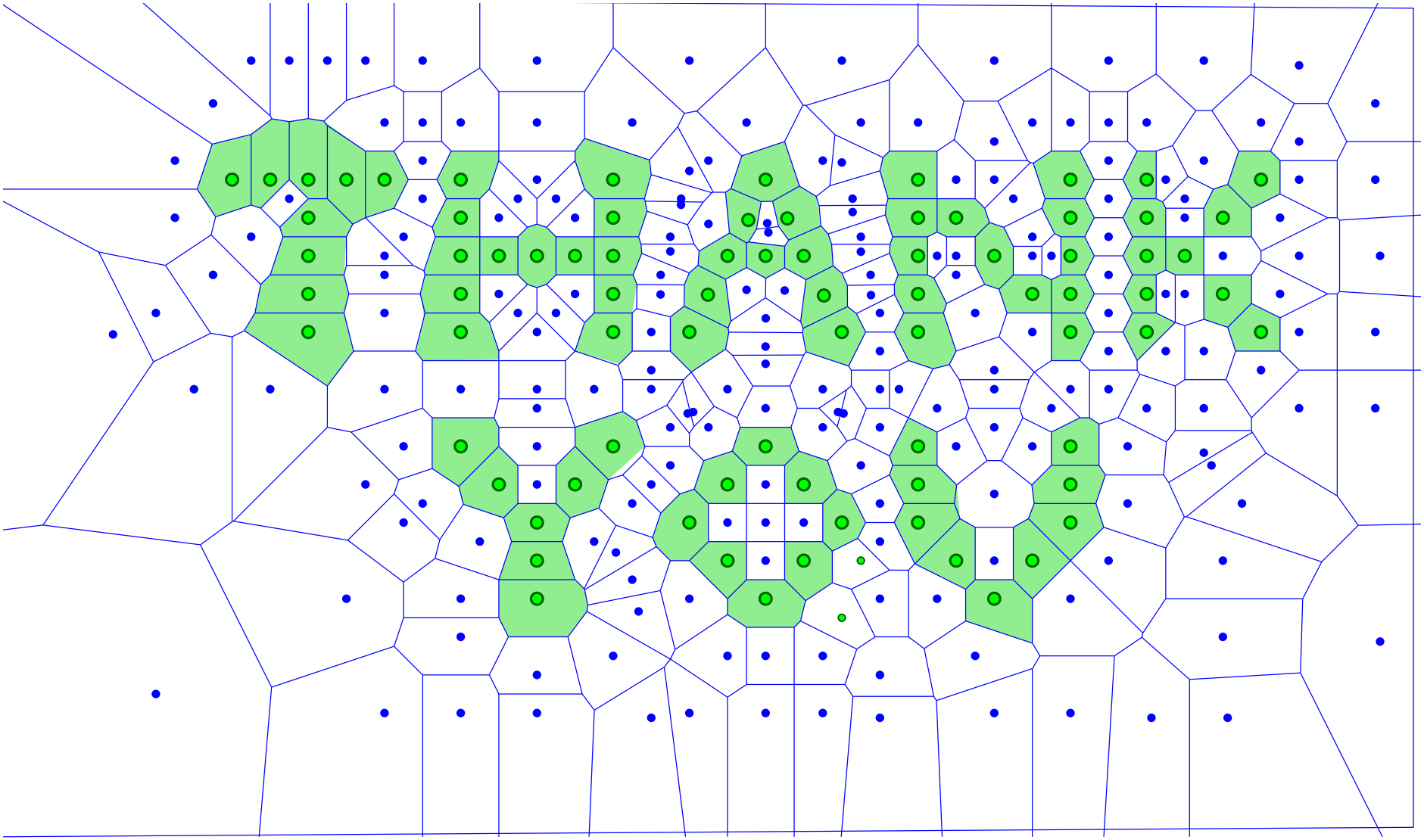

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