An annotated bibliography on 3SUM-hard problems

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The following articles contain examples of 3SUM-hard problems. Gajentaan and Overmars described a large class of 3SUM-hard problems in their paper *On a class of $O(n^2)$ problems in Computational Geometry.* These problems are at least as hard as the base problem 3SUM: *Given a set S of n integers, are there at least three elements in S that sum up to 0?* The best known algorithm for 3SUM takes $\Theta(n^2)$ time. Some other 3SUM-hard problems are:
3SUM' Given three sets of integers A, B and C of total size n, are there $a \in A, b \in B$ and $c \in C$ with $a + b = c$?
GEOMBASE Given a set of n points with integer coordinates on three horizontal lines $y = 0, y = 1$ and $y = 2$, determine whether there exists a non-horizontal line containing three of the points.
POINT-ON-3-LINES Given a set of n lines in the plane, is there a point that lies on at least three of them?
3-POINTS-ON-A-LINE Given a set of n points in the plane, is there a line that contains at least three of the points?

Mark de Berg, Marko de Groot and Mark H. Overmars. Perfect binary space partitions. In *Computational Geometry: Theory and Applications* 7:81 - 91, 1997.
The authors give a method to either construct a perfect binary space partition, or to decide that no perfect binary space partition exists for the arrangements of line segments, in $O(n^2 \log n)$ time. A space partition is called perfect when none of the objects is cut by the lines used by the BSP. They show that it at least as hard as the problem GEOMBASE to decide whether a set of n disjoint line segments in the plane admits a perfect binary space partition.
Instead of the points on three horizontal lines as in GEOMBASE, three parallel lines with holes are created. The holes are placed at the original locations of the points. This transformation can be done in time $O(n \log n)$. When there is a line through the points a, b and c on the horizontal lines, a seperator exists for the set of line segments that goes through the holes related to a, b and c. Given such a separator, a perfect BSP exists. They also prove the reverse; if a perfect BSP exists of the set of line segments constructed, then the first partition line must run through three holes in the horizontal lines.

Michael Soss, Jeff Erickson and Mark H. Overmars. Preprocessing Chains for Fast Dihedral Rotations Is Hard or Even Impossible. In *Computational Geometry: Theory and Applications* 26(3): 235 - 246, 2002.
A polymer can be modelled as a chain of line segments in three dimensions. A dihedral rotation moves the chain around one of the edges of these line segments. For example, a dihedral rotation at an edge $\overline{uv}$ rotates subchain A around subchain

B, keeping the angle between u and v fixed, where $u \in A$ and $v \in B$ Preprocessing a chain of n edges and performing n static dihedral rotation queries is 3SUM-hard. Preprocessing means checking whether a rotation can be performed without self-intersection of the polymer.
The authors create a polygonal chain consisting of an axis-parallel staircase(B) and two combs (A' and C'). For each element $a' \in A'$ the left comb contains a very narrow upward tooth centered on the line $x = a'$. For each element $c' \in C'$ the right comb contains a very narrow downward tooth centered on the line $x = c'$. For each element $b \in B$, the staircase contains a vertical edge on the line $x = -b/2$.
The rotation is performed at a vertex in the staircase in such a way the only place the polygon can self-intersect is at the teeth of the combs. The rotation causes two teeth to collide if and only if $a' = -c' - b$, or equivalently, $a' + b + c' = 0$. This is exactly the 3SUM base problem.

Jeff Erickson. New lower bounds for convex hull problems in odd dimensions. In *SIAM Journal on Computing* 28(4): 1198 - 1214, 1999.
Deciding whether a convex hulls in $\mathbb{R}^d$ is simplicial is $\lceil d/2 \rceil$SUM-hard. A convex hull is simplicial if all its facets are simplices. So in the fifth dimension detecting simplicial convex hulls is 3SUM-hard.
Given a set of integers $X = \{x_1, x_2, ..., x_n\}$, replace them with the larger set $X = \{x_1^\flat, x_1^\sharp, x_2^\flat, x_2^\sharp, ..., x_n^\flat, x_n^\sharp\}$, where $x_i^\flat = x_i - 2^{-1}$ and $x_i^\sharp = x_i + 2^{-i}$ for all i. Then consider the points $w_5(X)$ obtained by lifting X onto the weird moment curve in $\mathbb{R}^5$. The authors show that the convex hull is non-simplicial if and only if some three elements of X sum to zero, which is again the 3SUM base problem.

Jeff Erickson, Sariel Har-Peled and David M. Mount. On the least median square problem. In *Proceedings of the 20th Annual Symposium on Computational Geometry* 273 - 279, 2003.
Given a set of n points on the d-dimensional integer lattice Z^d, do any $d+1$ points lie on a common hyperplane? The d-dimensional affine degeneracy problem is $(d+1)$-SUM-hard. This is a generalisation of the 3-POINTS-ON-A-LINE problem in two dimensions.

Boris Aronov and Sariel Har-Peled. On Approximating the Depth and Related Problems. To appear in *Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms*, 2005.
Determining a point of maximum depth in an arrangement of n disks is 3SUM-hard. The authors reduce the POINT-ON-3-LINES problem to this problem in $O(n \log n)$ time.
Consider a set L of n lines. One can compute in $O(n \log n)$ time an axis-parallel square Q containing all vertices of the arrangement $\mathcal{A}(L)$. Furthermore, one can compute a lower bound Δ on the distance between a vertex of $\mathcal{A}$ and a line of L not passing through it. Next, each line $l \in L$ is replaced by two sufficiently large disks D_l, D'_l of equal radii, such that $\delta D_l \cap \delta D'_l = l \cap \delta Q$ (so that $D_l \cap D'_l$ is symmetric with respect to l) and $D_l \cap D'_l$ lies in the strip of half-width $\Delta/4$ centered at l. Let D be the resulting set of $2n$ disks. There is a point in Q contained in at least $n+3$ disks of D if and only if some three lines of L share a point.

Otfried Cheong, Alon Efrat and Sariel Har-Peled. On finding a guard

that sees most and a Shop that sells most. In *Proceedings of the 15th Annual ACM-SIAM Symposium on Discrete Algorithms* 1098 - 1107, 2004.
Approximating the largest visible polygon, measured in area, up to a constant factor is 3SUM-hard.
Given a polygon P, there is a constant $c > 0$ such that the $(c-1)$-approximating the largest visible polygon in P is 3SUM-hard.
The authors reduce the POINT-ON-3-LINES problem to this problem in $O(n \log n)$ time.
Let L be a set of n lines with integer coefficients. One can resize and translate L such that all the vertices of the arrangement of L lie in the unit square $[0.25, 0.75]^2$. Next, consider the axis parallel square S of side length M^{10} centered at the origin, and replace every line $l \in L$ by thickening it to a rectangle r_i such that the intersection of r_i with S is of area 2, where $M \leq n^{10}$ is an appropriate large number which is function of the input. Furthermore, all those rectangles are disjoint outside the unit square. Let R denote the resulting set of rectangles. Next, consider the polygon $P = (\cup_{r \in R} r) \cup [0,1]^2$. If there are three lines in L that pass through a common point, then there is a point that stabs three rectangles of R, and sees an area $\geq 3 \cdot 2 - o(1)$ inside P. Similarly, if there is no point that is contained in three lines of L, then clearly, every point inside P sees at most $2 \cdot 2 + 1 + o(1)$ area.

Gill Barequet and Sariel Har-Peled. Some variants of polygonal containment and minimum Hausdorff distance under translation are 3sum hard. In *Proceedings of the 10th Annual ACM-SIAM Symposium on Discrete Algorithms* 862 - 863, 1999.
Two 3SUM-hard problems are presented:
EQDIST (EQUAL DISTANCE): Given two sets P and Q of n and $m = O(n)$ real numbers, respectively, is there a pair $p_1, p_2 \in P$ such that $p_1 - p_2 = q_1 - q_2$?
SEGCONTPNT (SEGMENT CONTAINING POINTS): Given a set P of n real numbers and a set Q of $m = O(n)$ disjoint intervals of real numbers, is there a real number (translation) v such that $P + v \subseteq Q$?
The authors prove that 3SUM' $\lll_n$ EQDIST and 3SUM' $\lll_n$ SEGCONTPNT.
Let (A, B, C) be an instance of 3SUM', such that $|A| = |B| = |C| = n$. The authors assume that $A \cup B \cup C \subseteq (0,1)$. They create the set $P = \{100i, 100i + 3 - c_i | c_i \in C, i = 1, ..., n\}$ and define $Q = A \cup \{3 - b | b \in B\}$. (P, Q) is a corresponding instance of the EQDIST problem. If there is a solution for the 3SUM' instance, there is a solution for the EQDIST instance: $(100i + 3 - c_i) - (100i) = (3 - b) - a$. They show that the opposite also applies.
The EQDIST instance can easily be applied to an SEGCONTPNT instance by adding two new intervals to Q: $Q' = [-100(n-1), 94] \cup Q \cup [100, 100(n-1) + 6]$. The length and location of the two additional intervals are chosen in such a way that a segment can contain a translation of only $(n-1)$ pairs of P, and that even the union of the two intervals can not contain any translation of P. Thus, there exists a translation $P + v \subseteq Q'$ iff there are two points of $P + v$ that are covered by Q.

Prosenjit Bose, Marc van Kreveld and Godfried Toussaint. Filling polyhedral molds. In *Proceedings of the 3rd Workshop Algorithms Data Structures, volume 709 of Lecture Notes in Computer Science*, 210 - 221, 1993.
A polyhedron is 1-fillable if it can be filled through one pingate with one direction of gravity such that no air holes arise. The 1-fillability problem is a 3SUM-hard problem. The rectancle covering problem can be reduced to the 1-fillability prob-

lem in $O(n \log n)$ time.
Let I be an instance of the rectangle covering problem, i.e., given a set R of n rectangles in the plane and a rectangle $RECT$, decide if the union of the rectangles in R cover $RECT$. A polyhedron P is constructed in such a way that P is 1-fillable if and only if the rectangle $RECT$ is not covered by R.

Esther M. Arkin, Yi-Jen Chiang, Martin Held, Joseph S. B. Mitchell, Vera Sacristán, Steven S. Skiena and Tae-Cheon Yang. On Minimum-Area Hulls. In *Algorithmica* 21(1): 119 - 136, 1998.
Given an n-vertex simple polygon P, find a minimum-area, star-shaped polygon P^* containing P. When the vertices of P^* are constrained to be vertices of P, finding P^* is 3SUM-hard.
The problem is reduced to the POINT-ON-3-LINES problem in $O(n \log n)$ time.

Daniel Archambault, William Evans and David Kirkpatrick. Computing the set of all distant horizons of a terrain. In *Proceedings of the 16th Canadian Conference on Computational Geometry* 76 - 79, 2004.
The problem of computing all distant horizon edges, or even computing the cardinality of this set, is 3SUM-hard. The reduction to the GEOMBASE problem is similar to the reduction from GEOMBASE to the problem SEPARATOR1, as shown in the paper by Gajentaan and Overmars.

Manuel Abellanas, Ferran Hurtado, Christian Icking, Rolf Klein, Elmar Langetepe, Lihong Ma, Belén Palop and Vera Sacristán. Smallest color-spanning objects. In *Proceedings of the 9th European Symposium on Algorithms* 278 - 289, 2001.
In a set of colored points, computing the narrowest strip of arbitrary orientation that encloses at least one site of each color is 3SUM-hard. It can be solved in $O(n^2\alpha(k) \log k)$, where n is the number of sites and $k \leq n$ is the number of colors.