Qualifying Examination
Theoretical Computer Science
Friday, March 3, 2017

Part II: Automata and Complexity

Instructions:

1. This is a closed book exam.
2. The exam has four problems worth 25 points each. Read all the problems carefully to see the order in which you want to tackle them. You have all day (9am–5pm) to solve the problems.
3. Write clearly and concisely. You may appeal to some standard algorithms/facts from text books unless the problem explicitly asks for a proof of that fact or the details of that algorithm.
4. If you cannot solve a problem, to get partial credit write down your main idea/approach in a clear and concise way. For example you can obtain a solution assuming a clearly stated lemma that you believe should be true but cannot prove during the exam. However, please do not write down a laundry list of half baked ideas just to get partial credit.

May the force be with you.

Problem 1: 1-dimensional Sperner's problem is defined on a 1-dimensional grid on interval $[0, 2^n - 1]$, with each integer being a grid point. There are two colors red and blue represented by 0 and 1 bit respectively. There is a Boolean circuit named Color which outputs color (0/1 bit) of a grid point given its bit representation, with a guarantee that Color(0)=red, Color($2^n - 1$)=blue. We can show that there exists an integer $0 \leq k \leq 2^n - 1$ such that Color(k)=red and Color($k+1$)=blue, and it can be computed in $O(n)$ calls to the Color circuit by doing binary search on k.

Show that checking if there are more than one such ks is NP-hard. (Hint: Reduce from 3-SAT).

Problem 2: If $\mathcal{S} = \{S_1, \cdots, S_m\}$ is a collection of subsets of a finite set U, the VC dimension of $\mathcal{S}$, denoted $VC(\mathcal{S})$, is the size of the largest set $X \subseteq U$ such that for every $X' \subseteq X$, there is an i for which $S_i \cap X = X'$. (We say that X is shattered by S.)

A Boolean circuit C succinctly represents collection $\mathcal{S}$ if S_i consists exactly of those elements $x \in U$ for which $C(i, x) = 1$. Finally,

$$\text{VC-DIMENSION} = \{\langle C, k\rangle : C \text{ represents a collection } \mathcal{S} \text{ s .t. } VC(\mathcal{S}) \geq k\}$$

1. Show that VC-DIMENSION $\in \Sigma_3^P$.
2. Show that VC-DIMENSION is Σ_3^P - hard.

Problem 3: All Turing machines in this problem are deterministic, have a single two way infinite tape, have tape alphabet {0,1,blank}, and on each transition *must* move either left or right (they may not stay in the same cell). Are the following problems decidable?

1. Given as input the description of a Turing machine M, whether it ever makes a sequence of *three* consecutive left moves when run with no input (an all blank tape).
2. Given as input the description of a Turing machine M, whether it ever makes a sequence of *two* consecutive left moves when run with no input (an all blank tape).

Problem 4: Consider the 2-knapsack problem where we are given n items each with a non-negative size s_i and a profit p_i. We are also given 2 knapsacks with capacities B_1 and B_2 respectively. Goal is to find a feasible packing of items into knapsacks to maximize the profit. A packing is simply a partial assignment of items to knapsacks such that the total size of items assigned to a given knapsack does not violate its capacity. Show that there is no FPTAS (fully polynomial-time approximation scheme) for this problem unless $P = NP$. An FPTAS provides for each fixed $\epsilon > 0$ a $(1 - \epsilon)$ approximation in time polynomial in the input size *and* $1/\epsilon$.