Problem 1: 1-dimensional Sperner’s problem is defined on a 1-dimensional grid on interval [0, \(2^n - 1\)], with each integer being a grid point. There are two colors red and blue represented by 0 and 1 bit respectively. There is a Boolean circuit named COLOR which outputs color (0/1 bit) of a grid point given its bit representation, with a guarantee that COLOR(0)=red, COLOR(\(2^n - 1\))=blue. We can show that there exists an integer 0 \(\leq k \leq 2^n - 1\) such that COLOR(\(k\))=red and COLOR(\(k+1\))=blue, and it can be computed in \(O(n)\) calls to the COLOR circuit by doing binary search on \(k\).

Show that checking if there are more than one such \(k\)s is NP-hard. (Hint: Reduce from 3-SAT).
Problem 2: If \( S = \{S_1, \cdots, S_m\} \) is a collection of subsets of a finite set \( U \), the VC dimension of \( S \), denoted \( VC(S) \), is the size of the largest set \( X \subseteq U \) such that for every \( X' \subseteq X \), there is an \( i \) for which \( S_i \cap X = X' \). (We say that \( X \) is shattered by \( S \).)

A Boolean circuit \( C \) succinctly represents collection \( S \) if \( S_i \) consists exactly of those elements \( x \in U \) for which \( C(i, x) = 1 \). Finally,

\[
VC-DIMENSION = \{\langle C, k \rangle : C \text{ represents a collection } S \text{ s.t. } VC(S) \geq k\}
\]

1. Show that \( VC-DIMENSION \in \Sigma^P_3 \).
2. Show that \( VC-DIMENSION \) is \( \Sigma^P_3 \)-hard.

Problem 3: All Turing machines in this problem are deterministic, have a single two way infinite tape, have tape alphabet \( \{0,1,\text{blank}\} \), and on each transition must move either left or right (they may not stay in the same cell). Are the following problems decidable?

1. Given as input the description of a Turing machine \( M \), whether it ever makes a sequence of three consecutive left moves when run with no input (an all blank tape).
2. Given as input the description of a Turing machine \( M \), whether it ever makes a sequence of two consecutive left moves when run with no input (an all blank tape).

Problem 4: Consider the 2-knapsack problem where we are given \( n \) items each with a non-negative size \( s_i \) and a profit \( p_i \). We are also given 2 knapsacks with capacities \( B_1 \) and \( B_2 \) respectively. Goal is to find a feasible packing of items into knapsacks to maximize the profit. A packing is simply a partial assignment of items to knapsacks such that the total size of items assigned to a given knapsack does not violate its capacity. Show that there is no FPTAS (fully polynomial-time approximation scheme) for this problem unless \( P = NP \).

An FPTAS provides for each fixed \( \epsilon > 0 \) a \( (1 - \epsilon) \) approximation in time polynomial in the input size and \( 1/\epsilon \).