Qualifying Examination
Theoretical Computer Science
Thursday, October 18, 2018
Part I: Algorithms

Instructions:

1. This is a closed book exam.
2. The exam has four problems worth 25 points each. Read all the problems carefully to see the order in which you want to tackle them. You have all day (9am–5pm) to solve the problems.
3. Write clearly and concisely. You may appeal to some standard algorithms/facts from textbooks unless the problem explicitly asks for a proof of that fact or the details of that algorithm.
4. If you cannot solve a problem, to get partial credit write down your main idea/approach in a clear and concise way. For example you can obtain a solution assuming a clearly stated lemma that you believe should be true but cannot prove during the exam. However, please do not write down a laundry list of half-baked ideas just to get partial credit.
5. In problems with multiple parts, if you cannot solve a part, you can still solve subsequent parts (assuming that claims from earlier parts are true) and get partial credit.

May the force be with you.

Problem 1: Given a graph $G = (V, E)$, recall that a subset $S \subseteq V$ is a *dominating set* in G if for all $u \in V - S$, there is a neighbor v of u such that $v \in S$. Consider the following generalization: for a parameter $k \geq 1$, a subset $S \subseteq V$ is a *k-dominating set* if for all $u \in V - S$, there are k distinct neighbors of u in S. For any fixed constant k, describe a polynomial time algorithm for finding the minimum-cardinality k-dominating set in a given tree $T = (V, E_T)$.

Problem 2: A graph G is (α, β)-*separable*, for constants $\alpha > 0$ and $\beta \in (0, 1)$, if for *every* subgraph $H \subseteq G$, and *every* non-negative weighting $w : V(H) \to \Re^+$ of its vertices, there is a set $X \subseteq V(H)$, such that

- $|X| \leq \alpha |V(H)|^\beta$;
- $V(H) \setminus X$ is the disjoint union of two sets Y, Z, such that there is no edge between a vertex of Y and a vertex of Z in H;
- $w(Y) = \sum_{y \in Y} w(y) \leq (2/3) w\bigl(V(H)\bigr)$, and $w(Z) \leq (2/3) w\bigl(V(H)\bigr)$.

The set X is called a *separator*. Assume that one can compute such a separator in linear time. Such separators are known for many families of graphs, famously including planar graphs and trees.

In the following, let G be an (α, β)-separable graph with n vertices.

(a) Prove that G is $c_1(\alpha, \beta)$-degenerate, where $c_1(\alpha, \beta)$ denotes a constant that depends only on α and β. A graph G is c-*degenerate*, if for every subgraph H of G, H has a vertex of degree at most c.

(b) Prove that G has an independent set of size $\geq n/c_2$, where $c_2 = c_2(\alpha, \beta)$ is some constant.

(c) Given a parameter $\delta \in (0, 1)$, show that one can compute (efficiently) a set Y of vertices of size $\leq \delta n$, such that all the connected components of the graph $H = G - Y$ are of size at most $c_3 = c_3(\alpha, \beta)$, where $G - Y$ denotes the graph resulting from deleting the vertices of Y from G.

(d) Describe a PTAS for computing the largest independent set in G. Formally, for a fixed $\delta \in (0, 1)$, give a polynomial time algorithm that outputs an independent set in G of size $\geq (1 - \delta)\mathrm{opt}$, where opt is the size of the largest independent set in G.

The calculations involved in the above are not pretty – there is no need to be too careful or too precise about the values of the constants mentioned above.

Problem 3: Let G be an undirected graph with n vertices and m edges that has a vertex cover of size k (for simplicity, assume that k is given).

(a) Present an algorithm for computing a 2-approximation of the minimum vertex cover of G. What is the running time of your algorithm?

(b) Provide an algorithm that computes the minimum vertex cover of G in $O(2^{2k}(n + m))$ time. (Hint: Think about a maximal matching in G and how it interacts with the optimal vertex cover.)

(c) Prove that any vertex of degree strictly larger than k must be in the optimal vertex cover.

(d) Provide an algorithm that computes the minimum vertex cover of G in $O(n+m+2^{2k}k^c)$ time, where c is some constant.

Problem 4: Let $G = (V, E)$ be an undirected simple graph (with no self-loops and multiedges). A node $u \in V$ is a *cut-vertex* of G if $G - u$ has at least two non-trivial connected components. We say that G is a *block* if G has no cut-vertices. Note that K_2 is a block. G is a *non-trivial* block if $|V| \geq 3$.

(a) Prove that in any non-trivial block, for any two nodes u, v there is a cycle that contains u and v.

(b) Prove that in any non-trivial block, for any two edges e, e' there is a cycle that contains e and e'. [Hint: use (a) on a modified graph.]