It is a long way to get accepted.

1.A. [30 Points] Given $k$ DFAs $M_1, \ldots, M_k$ each of at most $n$ states, show that if the intersection of their $k$ corresponding languages $L = L(M_1) \cap \cdots \cap L(M_k)$ is nonempty, then $L$ must contain a string of length at most $n^k$.

1.B. Prove that the following problem is in $\text{PSPACE}$: given an integer $k$ and $k$ DFAs $M_1, \ldots, M_k$, decide whether $L(M_1) \cap \cdots \cap L(M_k)$ is empty.

[Note: $k$ is not a constant! Hint: Savitch’s theorem, $\text{PSPACE} = \text{NPSPACE}$, may be useful.]

A language $S \subseteq \{0, 1\}^*$ is sparse if there is some polynomial $p(n)$ such that for all $n \geq 0$ that $S \cap \{0, 1\}^n \leq p(n)$. For a language $L$, show that $L \in \text{P/poly}$ iff $L \in \text{P}^S$ for some sparse language $S$.

Recall the model of interactive proofs, where a computationally unbounded prover $P$ interacts with a polynomial-time probabilistic verifier $V$. Consider the complexity class of languages with efficient interactive proofs where we consider different completeness and soundness parameters. That is, define $\text{IP}_{c,s}$ to be those languages $L$ where

$$x \in L \implies \exists P \Pr_V[(P \leftrightarrow V)(x) \text{ accepts}] \geq c$$

$$x \notin L \implies \forall P \Pr_V[(P \leftrightarrow V)(x) \text{ accepts}] \leq s.$$ 

The standard definition of the complexity class of interactive proofs is given by $\text{IP} := \text{IP}_{\frac{1}{2}, \frac{1}{2}}$.

3.A. Give a direct proof that $\text{IP}_{\frac{2}{3}, \frac{1}{3}}^n = \text{IP}_{1-\frac{n}{2}, \frac{1}{2}+\frac{n}{2}}$, where $n = |x|$.

3.B. Prove that $\text{IP}_{\frac{2}{3}, \frac{1}{3}} = \text{IP}_{1, \frac{1}{3}}$.

3.C. Prove that $\text{IP}_{\frac{1}{2}+\frac{n}{2}, \frac{1}{2}} = \text{IP}_{\frac{3}{2}, \frac{1}{3}}^n$.

$k\text{UP}$ are problems solvable in $\text{NP}$ by a machine that has at most $k$ accepting computations on any input. We will denote $1\text{UP}$ as simply $\text{UP}$. It is clear that $\text{P} \subseteq k\text{UP} \subseteq (k + 1)\text{UP}$ for each $k$, but not known if the hierarchy is strict. Prove that, $\text{P} = \text{UP}$ iff $\text{P} = 2\text{UP}$.

Aside: This observation can be coupled with induction to conclude that $\text{P} = k\text{UP}$ for some $k$ iff $\text{P} = k\text{UP}$ for all $k$. Please do not prove this generalization.