University of Illinois at Urbana-Champaign
Department of Computer Science

Qualifying Examination—Part II

Theoretical Computer Science

9:00–12 noon, Thursday, October 19, 1995
3211 Digital Computer Laboratory

Do not write your name anywhere on this examination, so that the exam can be graded without knowledge of your identity. Write your ID number legibly in the box below and at the upper right corner of every page.

ID Number:
Pseudonym:

Do all three problems in this booklet. All problems yesterday and today are equally weighted, so do not spend too much time on any one question. Blank pages follow each problem for extra workspace.

4. **Graphs and algorithms.** Let $G = (V, E)$ be an undirected simple graph with $|V|$ vertices and $|E|$ edges. The *average degree* of a vertex in V is

$$A(G) = \frac{1}{|V|} \sum_{u \in V} \deg_G(u),$$

where $\deg_G(u)$ is the number of edges in E incident to u. A graph $H = (W, F)$ is a *subgraph* of G if $W \subseteq V$ and $F \subseteq E$.

(a) Prove there exists a subgraph $H = (W, F)$ of G with $|F| \geq |E|/3$ and $\deg_H(u) \geq A(G)/3$ for *every* $u \in W$.

(b) Design a polynomial-time algorithm that constructs a subgraph H satisfying the conditions in (a). Analyze your algorithm.

(c) Describe a data structure for G and an algorithm based on this data structure that constructs H in time $O(|V| + |E|)$.

Since an answer to (c) implies an answer to (b) you may choose to skip (b). You get credit for (b) if you answer (c) correctly.

5. **Algorithm design and analysis.** *Arbitrage* is the use of discrepancies in currency exchange rates to make a profit. For example, the may be a small period of time during which US$1 buys £0.75 and £1 buys A$2 Australian dollars, and one Australian dollar buys US$0.70. Thus a smart trader can trade one U.S. dollar and end up with $0.75 \times 2 \times 0.70 = 1.05$ U.S. dollars, a profit of 5%. Suppose there are n currencies, $1, 2, \ldots, n$, and for each ordered pair $\langle i, j\rangle$ there is a number R_{ij} that gives the exchange rate between currency i and currency j, that is, one unit of currency i buys R_{ij} units of currency j.

   (a) Devise and analyse an algorithm to determine, given a set of currency exchange rates $\{R_{ij}\}$, whether or not there is a way to make a profit via some sequence of exchanges.

   (b) Either extend your algorithm above to find the maximum profit, or argue that determining the maximum profit possible is NP-hard. Note: If there is a way to make a profit, one can repeat the process to obtain an arbitrarily large profit. For example, in the case above, we could achieve $(1.05)^2$ growth by taking the US dollars and repeating the process. By "maximum profit" we mean the most one can make via a sequence of exchanges such that each currency is bought at most one time, and sold at most one time.

6. **Automata.**

In this problem we define a notion of acceptance that is dual to nondeterministic acceptance. An ∀FA is like an NFA except that it accepts an input x iff all computations on input x are accepting. This model has an interesting application to a two-person game (see below).

(a) Briefly show how to convert an ∀FA to an equivalent DFA.

(b) Lucy and Charlie are sitting at a table. On the table is a square tray with four glasses at the corners. Charlie's goal is to turn all the glasses either right-side up or upside down. However, Charlie is blindfolded and he is wearing mittens. He does not know the intitial state of the glasses. If they are initially all turned the same way, then Charlie automatically wins. In his turn, Charlie may grab one or two glasses and turn them over; however, because of the blindfold and mittens he cannot see or feel whether the glasses he grabbed are right-side up or upside down. He can, however, choose whether to grab adjacent glasses or diagonally opposite glasses or just a single glass. If the glasses are all turned the same direction, Lucy announces that Charlie has won. Otherwise, Lucy may rotate the tray, just to make Charlies's goal harder.

Find the shortest sequence of actions by Charlie that is guaranteed to win the game, no matter how Lucy plays. Prove that your solution is correct and is the shortest possible.