Spring 2022 – Theory qual part II: Automata & Complexity

1 Small circuits and formulas.

In this problem you will prove upper bounds for the maximal circuit complexity of a boolean function $f : \{0,1\}^n \to \{0,1\}$ over the AND, OR, and NOT basis of gates.

1.A. Show f has an $O(n2^n)$-size formula of *constant* depth.

1.B. Show f has an $O(2^n)$-size formula.
Hint: Use recursion.

1.C. Show f has an $O\left(\frac{1}{n}2^n\right)$-size *circuit*.
Hint: Apply memoization to the recursion from part (**1.B.**). Do not solve the same problem multiple times. Exploit the expressiveness of *circuits*, rather than just formulas.
Hint: Perhaps try applying your solution to (**1.B.**) to the parity of three bits $x \oplus y \oplus z$, and see how one can memoize the recursion.

2 You reach.

The decision problem UReach is to decide, given an undirected graph G and two vertices a and b in G, whether there is a path in G from a to b. It is known that UReach is in the complexity class L.

Based on the above information, for each of the following statements, state whether it is true, false, or unknown. In each case, give justification for your answer and in the case where the truth of the statement is unknown, state any implications that might follow from it being true or false.

2.A. Let us say that a nondeterministic Turing machine M is symmetric if for any two configurations c_1 and c_2 of M, if $c_1 \to_M c_2$, then $c_2 \to_M c_1$. We write SL for the class of all languages that are accepted by a symmetric Turing machine using $O(\log n)$ work space on inputs of length n. Then, $\mathsf{SL} \subseteq \mathsf{L}$.

2.B. If UReach is complete for P, then $\mathsf{NL} = \mathsf{NP}$.

3 OV and SETH. (Fall 2018.)

In the *Orthogonal Vectors (OV)* problem, we are given two sets of n d-dimensional 0-1 vectors $A \subseteq \{0,1\}^d$ and $B \subseteq \{0,1\}^d$ with $|A| = |B| = n$, and we want to decide whether there are vectors $\vec{a} \in A$ and $\vec{b} \in B$ that are orthogonal, i.e., such that $\langle \vec{a}, \vec{b} \rangle = \sum_i a_i b_i = 0$. The *Orthogonal Vectors Conjecture (OVC)* of fine-grained complexity states that for every $\delta > 0$ there is a $c \geq 1$ such that OV cannot be solved in $n^{2-\delta}$ time on instances with $d = c \cdot \lg n$. This problem will explore OVC and how it relates to other conjectures in fine-grained complexity.

In particular, one formulation of the *Strong Exponential Time Hypothesis (SETH)* states that for every $\epsilon > 0$ there is a k and a c such that k-SAT for n variables and cn clauses requires time $\Omega((2-\epsilon)^n)$. (The input to k-SAT is a CNF formula where clauses have length k.)

3.A. Show that OV can be solved in $O(n^2 \mathsf{poly}(d))$ time.

3.B. Show that OV can be solved in $O(n 2^{O(d)})$ time.

3.C. Show a reduction from k-SAT with n variables and m clauses to OV with $O(2^{n/2})$ vectors in m dimensions, that runs in time $O(2^{n/2}\mathsf{poly}(m))$. [Hint: partition the n variables into two groups of size $\frac{n}{2}$ each.]

3.D. Conclude that SETH implies OVC.

4 Some TMs just want to be accepted. (Spring 2016)

Consider the following two definitons of log-space counting problems. A function $f : \{0,1\}^* \to \mathbb{N}$ is in $\#L_a$ if there is a non-deterministic Turing machine M_f that on input x of length n uses $O(\log n)$ space and is such that the number of accepting paths of $M_f(x)$ equals $f(x)$. Such a function is in $\#L_b$ if there is a relation $R(\cdot,\cdot)$ that is decidable in log-space, and a polynomial p such that if $R(x,y)$ then $|y| \leq p(|x|)$ and $f(x)$ equals $|\{y|R(x,y)\}|$.

4.A. Prove that all functions in $\#L_a$ can be computed in polynomial time.

4.B. Prove that $\#L_b$ equals $\#P$. Recall that $\#P$ is the class of functions $f : \{0,1\}^* \to \mathbb{N}$ such that there is a non-deterministic polynomial TM M_f such for all x, $f(x)$ equals the number of accepting computation paths of M_f on x.