Homework 1, Due 23:59:59, 9/26/10 - slide it under the door of my office (SC 3306)

CS 598shp - Randomized Algorithms - Fall 2010

September 23, 2010

Version: 1.4

1 General Policy

Every one should submit the homework on their own. Speaking with other people is encouraged if you are really stuck, but you should probably try to do as much of it as possible on your own, if you get a considerable information from somebody you should mention it in your solution (which is quite alright). As for google policy, a lot of homework questions might have solutions on the web (hopefully not, but I can not be sure). I would *prefer* if you try to solve them without searching on the web, but if you use solutions found on the web, you should **explicitly** say so in your solution. In any case, I expect you to write your solution on your own (i.e., cut and paste is not acceptable), since the homeworks are there to demonstrate that you know the material and understand it.

Since I have to grade the homeworks myself, it would be so very nice of you to type the homeworks using latex (if you never used latex, and want some advice, come and speak with me). If you handwrite them, please use clear handwriting. Solution written by hand using unreadable script would result in a voodoo doll of the relevant person being sent to Guantanamo Bay. (In short, make an effort to write very clearly.)

If you have any questions, feel free to email me.

2 Problems

1. Cuts and stuff.
   - (A) Consider a graph $G = (V, E)$ with n vertices, m edges and a min cut of size k. Let $\mathcal{F}$ be the collection of all min-cuts in G (i.e., all the cuts in $\mathcal{F}$ are of size k). What is the probability that **MinCut** (the simpler variant – see class notes) would output a specific min-cut $S \in \mathcal{F}$?
     (Here, we are looking for a lower bound on the probability, since two minimum cuts of the same size might have different probabilities to be output.)
     [And yes, this is easy.]
   - (B) Bound the size of $\mathcal{F}$.

(C) A ***good cut*** is a cut $(S, \overline{S})$ such that the induced subgraphs G_S and $G_{\overline{S}}$ are connected.
Consider a specific good cut $C = \left(S, \overline{S}\right)$ with kt edges in it, where $t \geq 1$. What is the probability that **MinCut** would output this cut? (Again, proof a lower bound on this probability.)

(D) Consider the set $\mathcal{F}(t)$ of all good cuts in G of size at most kt. Bound the size of $\mathcal{F}(t)$.

(E) (Extra credit.) Consider the set $\mathcal{F}'(t)$ of All cuts in G (not necessarily all good) of size at most kt. Bound the size of $\mathcal{F}'(t)$.

2. Easier than it looks.

(A) Prove that $\displaystyle\sum_{i=0}^{n-k} \binom{2n}{i} \leq \frac{n}{4k^2} 2^{2n}$.
Hint: Consider flipping a coin $2n$ times. Write down explicitly the probability of this coin to have at most $n-k$ heads, and use Chebyshev inequality.

(B) Using (A), prove that $\binom{2n}{n} \geq 2^{2n}/4\sqrt{n}$ (which is a pretty good estimate).

(C) Prove that $\displaystyle\binom{2n}{n+i+1} = \left(1 - \frac{2i+1}{n+i+1}\right)\binom{2n}{n+i}$.

(D) Prove that $\displaystyle\binom{2n}{n+i} \leq \exp\left(\frac{-i(i-1)}{2n}\right)\binom{2n}{n}$.

(E) Prove that $\displaystyle\binom{2n}{n+i} \geq \exp\left(-\frac{8i^2}{n}\right)\binom{2n}{n}$.

(F) Using the above, prove that $\displaystyle\binom{2n}{n} \leq c\frac{2^{2n}}{\sqrt{n}}$ for some constant c (I got $c = 0.824...$ but any reasonable constant will do).

(G) Using the above, prove that

$$\sum_{i=t\sqrt{n}+1}^{(t+1)\sqrt{n}} \binom{2n}{n-i} \leq c 2^{2n} \exp\left(-t^2/2\right).$$

In particular, conclude that the probability of getting $2n$ coin flips to get less than $n - t\sqrt{n}$ heads (for t an integer) is smaller than $c' \exp(-t^2/2)$, for some constant c'.

(H) Let X be the number of heads in $2n$ coin flips. Prove that for any integer $t > 0$ and any $\delta > 0$ sufficiently small, it holds that $\mathbf{Pr}[X < (1-\delta)n] \geq \exp(-c''\delta^2 n)$, where c'' is some constant. Namely, the Chernoff inequality is tight in the worst case.

3. What happens on Uranus stays on Uranus.

There are currently k Uranusians alive on Uranus (they look a lot like cucumbers, but with legs). Each month exactly one of the Uranusians undergoes a critical event. Either it dies with probability p or it splits into two new Uranusians with probability p. With probability $1-2p$ nothing happens.

(A) Let X_i be the size of the population of the Uranus population after the ith month. What is $\mathbf{E}[X_i]$ and $\mathbf{V}[X_i]$? Here, you need to only provide a good upper bound on $\mathbf{V}[X_i]$, as computing it exactly seems hard.

(B) Let P_i be the probability that the population of Uranus is non-zero after i months. Prove that $\lim_{i=0}^{\infty} P_i = 0$. (There is a short and elegant argument showing that, but it is in fact not easy to see.)

(C) (Harder.) Let X be the number of months till the population of Uranus is extinct. Prove that $\mathbf{E}[X]$ is unbounded.