Homework 1, Due 23:59:59, 2/10/2014 - slide it under the door of my office (SC 3306)

CS 574 - Randomized Algorithms - Spring 2014

January 26, 2014

Version: 0.9

1. General Policy

The homeworks can be submitted in groups of size up to four – please submit a single copy, with the names/netid printed in the top of the solution. The solution should be typed using latex, and both a pdf and tex file should be submitted electronically (the details are TBD), in addition to the printed copy.

If you get a considerable amount of information from somebody you should mention it in your solution (which is quite alright). As for google policy, a lot of homework questions might have solutions on the web (hopefully not, but I can not be sure). I would *prefer* if you try to solve them without searching on the web, but if you use solutions found on the web, you should **explicitly** say so in your solution (and provide the link!). In any case, I expect you to write your solution on your own (i.e., cut and paste is not acceptable), since the homeworks are there to demonstrate that you know the material and understand it.

If you have any questions, feel free to email me.

2. Problems

1. Some probability required. **[20 Points]**

(A) **[5 Points]** Let X and Y be two random variables, such that $X, Y \in \{1, \ldots, 10\}$. Prove that $\mathbf{E}\Big[\mathbf{E}\big[X \big| Y\big]\Big] = \mathbf{E}\big[X\big]$ (note that X and Y are not necessarily independent).

(B) **[15 Points]** Let π be a random permutation of $\{1, \ldots, n\}$, and let $\mathcal{E}_i$ be the event that $\pi_i = \min(\pi_1, \ldots, \pi_i)$, and let X_i be its indicator variable.

(B.1) **[5 Points]** Prove that $\mathbf{E}[\sum_i X_i] = O(\log n)$.

(B.2) **[5 Points]** Prove that X_i and X_{i-1} are random independent variables.

(B.3) **[5 Points]** Let $X_{i_1}, \ldots, X_{i_k}$ be subset of the variables of $\{X_1, \ldots, X_n\}$ that does not contain X_i. Prove that $\mathbf{Pr}\Big[X_i = x \;\Big|\; X_{i_1}, \ldots, X_{i_k}\Big] = \mathbf{Pr}[X_i = x]$.

That is X_i is independent of any subset of the other variables. Why doesn't (B.2) implies this claim?

2. Cuts and stuff. **[30 Points]**

(A) **[5 Points]** Consider a graph $G = (V, E)$ with n vertices, m edges and a min cut of size k. Let $\mathcal{F}$ be the collection of all min-cuts in G (i.e., all the cuts in $\mathcal{F}$ are of size k). What is the probability that **MinCut** (the simpler variant – see class notes) would output a specific min-cut $S \in \mathcal{F}$?
(Here, we are looking for a lower bound on the probability, since two minimum cuts of the same size might have different probabilities to be output.)
[And yes, this is easy.]

(B) **[5 Points]** Bound the size of $\mathcal{F}$.

(C) **[5 Points]** A ***good cut*** is a cut $(S, \overline{S})$ such that the induced subgraphs G_S and $G_{\overline{S}}$ are connected.
Consider a specific good cut $C = \left(S, \overline{S}\right)$ with kt edges in it, where $t \geq 1$. What is the probability that **MinCut** would output this cut? (Again, proof a lower bound on this probability.)

(D) **[10 Points]** Consider the set $\mathcal{F}(t)$ of all good cuts in G of size at most kt. Bound the size of $\mathcal{F}(t)$.

(E) **[5 Points]** Consider the set $\mathcal{F}'(t)$ of all cuts in G (not necessarily all good) of size at most kt. Bound the size of $\mathcal{F}'(t)$.

3. Good betting strategy? **[50 Points]**

Consider the game where a player starts with $Y_0 = 1$ dollars. At every round, the player can bet a certain amount x (fractions are fine). With probability half she loses her bet, and with probability half she gains an amount equal to her bet. The game lasts for n rounds. The player is not allowed to go all in – because if she looses then the game is over. So it is natural to ask what her optimal betting strategy is, such that in the end of the game she has as much money as possible.

So, let Y_{i-1} be the money the player has in the end of the $(i-1)$th round, and she bets an amount $\psi_i \leq Y_{i-1}$ in the ith round. As such, in the end of the ith round, the amount of money she has is

$$Y_i = \begin{cases} Y_{i-1} - \psi_i & \text{LOSE: probability half} \\ Y_{i-1} + \psi_i & \text{WIN: probability half.} \end{cases}$$

(A) **[10 Points]** Prove that $\mathbf{E}[Y_i] = 1$, for all i. (Hint: Use other questions on this homework.)

(B) **[40 Points]** Part (A) implies that the game is pointless – in expectation the player does not gain anything. However, assume that the player is somewhat more lucky than average – she knows that she can win, say, $(1+\delta)/2$ fraction of her bets (instead of the predicted $1/2$) – for example, if the bets are in the stock market, she can improve her chances by doing more research on the companies she is investing in[1]. Here δ is some fixed parameter in the range $(0, 1/4)$, say.

[1] "I am a great believer in luck, and I find the harder I work, the more I have of it." – Thomas Jefferson.

Unfortunately, the player does not know which rounds she is going to be lucky in – so she still needs to be very careful.

Describe a betting strategy such that the lucky player ends up with as much money as possible, as a function of δ and n. Describe your betting strategy in detail, and prove a lower bound, as high as possible, on the amount of money the player ends up with. Your lower bound should be as simple as possible – so try to simplify it by doing some (mathematically) correct approximation, and in particular your bound should not contain any product ($\prod$), summation ($\sum$), or dots ($\ldots, \cdots$) in it. In short – make it as simple as possible while still being a meaningful bound.

This is a pretty open ended question, so decide in advance how much time you are willing to spend on it, and do not spend more time on this than that. In particular, finding the optimal betting strategy is probably extremely hard to impossible.

Hint: The lower bound I have in mind is surprisingly large.