Homework 2, Due 23:59:59, 3/3/2014 - slide it under the door of my office (SC 3306)

CS 574 - Randomized Algorithms - Spring 2014

March 4, 2014

Version: 0.9

General Policy. See previous homework.

1. Problems

1. Balanced occupancy.
For the occupancy problem, show that for $m = cn \ln n$, with probability at least $1 - o(1)$, every bin has at most $O(\log n)$ balls in it. Make c as small as possible – what is the value of c for your bound?

2. Is the Markov inequality tight?

(A) For an integer k, define a non-negative random variable X_k, such that $\mathbf{E}[X_k] = 1$, and $\mathbf{Pr}[X_k \geq k] = 1/k$. Namely, Markov's inequality can be tight for any k.

(B) Consider a positive integral random variable X with $\Delta = \mathbf{E}[X]$, Furthermore, for any number x, there exists an integer $y > x$, such that we have $\mathbf{Pr}\left[X \geq y\Delta\right] \geq \frac{c}{y}$, where $c > 0$ is some arbitrary constant.

Prove, that no such random variable X exists. Namely, Markov's inequality can not hold for too many times for a random variable.

3. The exploding occupancy problem.
You are throwing balls into n bins. If a bin is not empty, you empty the bin (since the bin exploded from having too many balls in it). Namely, as soon as a bin has two or more balls, you throw away these balls and empty the bin (i.e., at any points in time, a bin is either empty or contains a single ball).

(A) Prove that if you thrown in $n/4$ balls, then in expectation, at least $n/8$ bins are non empty in the end of throwing the balls.

(B) Prove that after $O(n^2)$ balls thrown into the bins, at some point there were (at least) $n/2$ non-empty bins, and this holds with probability, say, larger than $3/4$.

(Hint: Let X_i be the number of non-empty bins in the ith iteration. Define a variable Y_i (somehow) such that $X_i \geq Y_i$, then argue that for i sufficiently large, $Y_i \geq n/2$ with the required probability.)

4. Chernoff is tight by direct calculations.

For this question use only basic argumentation – do not use Stirling's formula, Chernoff inequality or any similar heavy machinery.

(A) Prove that $\sum_{i=0}^{n-k} \binom{2n}{i} \leq \frac{n}{4k^2} 2^{2n}$.

Hint: Consider flipping a coin $2n$ times. Write down explicitly the probability of this coin to have at most $n-k$ heads, and use Chebyshev inequality.

(B) Using (A), prove that $\binom{2n}{n} \geq 2^{2n}/4\sqrt{n}$ (which is a pretty good estimate).

(C) Prove that $\binom{2n}{n+i+1} = \left(1 - \frac{2i+1}{n+i+1}\right)\binom{2n}{n+i}$.

(D) Prove that $\binom{2n}{n+i} \leq \exp\left(\frac{-i(i-1)}{2n}\right)\binom{2n}{n}$.

(E) Prove that $\binom{2n}{n+i} \geq \exp\left(-\frac{8i^2}{n}\right)\binom{2n}{n}$.

(F) Using the above, prove that $\binom{2n}{n} \leq c\frac{2^{2n}}{\sqrt{n}}$ for some constant c (I got $c = 0.824...$ but any reasonable constant will do).

(G) Using the above, prove that

$$\sum_{i=t\sqrt{n}+1}^{(t+1)\sqrt{n}} \binom{2n}{n-i} \leq c2^{2n}\exp\left(-t^2/2\right).$$

In particular, conclude that when flipping fair coin $2n$ times, the probability to get less than $n - t\sqrt{n}$ heads (for t an integer) is smaller than $c' \exp(-t^2/2)$, for some constant c'.

(H) Let X be the number of heads in $2n$ coin flips. Prove that for any fixed $\delta > 0$ sufficiently small, there exists an integer n_δ, such that for any integer $n > n_\delta$, it holds that $\mathbf{Pr}[X < (1-\delta)n] \geq \exp(-c''\delta^2 n)$, where c'' is some constant. Namely, the Chernoff inequality is tight in the worst case.

5. Allergic reaction.

There are currently k bacteria[1] in your, say, peaceful school project. Each minute exactly one of the bacterium undergoes a critical event – it dies with probability 1/2, or it splits into two new bacteria with probability 1/2.

(A) Let X_i be the size of the population of the bacteria after the ith month. What is $\mathbf{E}[X_i]$ and $\mathbf{V}[X_i]$? Here, you need to only provide a good upper bound on $\mathbf{V}[X_i]$, as computing it exactly seems hard.

(B) Let P_i be the probability that the population of bacteria is non-zero after i months. Prove that $\lim_{i=0}^{\infty} P_i = 0$. (There is a short and elegant argument showing that, but it is not easy to see.)

(C) (Harder.) Let X be the number of months till the population of bacteria is extinct. Prove that $\mathbf{E}[X]$ is unbounded.

[1] The singular form is bacterium.