

# CS 573: Algorithms, Fall 2013

## Homework 4, due Monday, November 11, 23:59:59, 2013

### Version 1.0

Neatly print your name(s), NetID(s) on each submitted question. Remember that you have to submit each question on a separate page. Each student should submit his own homework. If you are on campus, submit the homework by submitting it in the homework boxes in the basement of SC.

Shortly after the celebration of the four thousandth anniversary of the opening of space, Angary J. Gustible discovered Gustible's planet. The discovery turned out to be a tragic mistake.

Gustible's planet was inhabited by highly intelligent life forms. They had moderate telepathic powers. They immediately mind-read Angary J. Gustible's entire mind and life history, and embarrassed him very deeply by making up an opera concerning his recent divorce.

– From Gustible's Planet, Cordwainer Smith

## Required Problems

### 1. Tedious Computations (40 PTS.)

(A) (10 PTS.) Let  $L$  be a linear program given in slack form, with  $n$  nonbasic variables  $N$ , and  $m$  basic variables  $B$ . Let  $N'$  and  $B'$  be a different partition of  $N \cup B$ , such that  $|N' \cup B'| = |N \cup B|$ . Show a polynomial time algorithm that computes an equivalent slack form that has  $N'$  as the nonbasic variables and  $B'$  as the basic variables. How fast is your algorithm?

(B) (3 PTS.) You are given a weighted, directed graph  $G = (V, E)$ , with weight function  $w : E \rightarrow \mathcal{R}$  mapping edges to real-valued weights, a source vertex  $s$ , and a destination vertex  $t$ . Show how to compute the value  $d[t]$ , which is the weight of the shortest weighted path from  $s$  to  $t$ , by using linear programming.

(C) (3 PTS.)

Given a graph  $G$  as in (A), write a single linear program that by solving it you compute  $d[v]$ , which is the shortest-path weight from  $s$  to  $v$ , for all the vertices  $v \in V$ .

(D) (4 PTS.)

In the *minimum-cost multicommodity-flow problem*, we are given a directed graph  $G = (V, E)$ , in which each edge  $(u, v) \in E$  has a nonnegative capacity  $c(u, v) \geq 0$  and a cost  $\alpha(u, v)$ . As in the multicommodity-flow problem (Chapter 29.2, CLRS), we are given  $k$  different commodities,  $K_1, K_2, \dots, K_k$ , where commodity  $i$  is specified by the triple  $K_i = (s_i, t_i, d_i)$ . Here  $s_i$  is the source of commodity  $i$ ,  $t_i$  is the sink of commodity  $i$ , and  $d_i$  is the demand, which is the desired flow value for commodity  $i$  from  $s_i$  to  $t_i$ . We define a flow for commodity  $i$ , denoted by  $f_i$ , (so that  $f_i(u, v)$  is the flow of commodity  $i$  from vertex  $u$  to vertex  $v$ ) to be a real-valued function that satisfies the flow-conservation, skew-symmetry, and capacity constraints. We now define  $f(u, v)$ , the *aggregate flow* to be sum of the various commodity flows, so that  $f(u, v) = \sum_{i=1}^k f_i(u, v)$ . The aggregate flow on edge  $(u, v)$  must be no more than the capacity of edge  $(u, v)$ .

The cost of a flow is  $\sum_{u,v \in V} f(u, v)\alpha(u, v)$ , and the goal is to find the feasible flow of minimum cost. Express this problem as a linear program.

(E) (5 PTS.)

Provide *detailed* solutions for the following problems, showing each pivoting stage separately.

$$\text{maximize } 6x_1 + 8x_2 + 5x_3 + 9x_4$$

subject to

$$2x_1 + x_2 + x_3 + 3x_4 \leq 5$$

$$x_1 + 3x_2 + x_3 + 2x_4 \leq 3$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

(F) (5 PTS.)

maximize  $2x_1 + x_2$

subject to

$$2x_1 + x_2 \leq 4$$

$$2x_1 + 3x_2 \leq 3$$

$$4x_1 + x_2 \leq 5$$

$$x_1 + 5x_2 \leq 1$$

$$x_1, x_2 \geq 0.$$

(G) (5 PTS.)

maximize  $6x_1 + 8x_2 + 5x_3 + 9x_4$

subject to

$$x_1 + x_2 + x_3 + x_4 = 1$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

(H) (5 PTS.)

minimize  $x_{12} + 8x_{13} + 9x_{14} + 2x_{23} + 7x_{24} + 3x_{34}$

subject to

$$x_{12} + x_{13} + x_{14} \geq 1$$

$$-x_{12} + x_{23} + x_{24} = 0$$

$$-x_{13} - x_{23} + x_{34} = 0$$

$$x_{14} + x_{24} + x_{34} \leq 1$$

$$x_{12}, x_{13}, \dots, x_{34} \geq 0.$$

## 2. Linear programming (30 PTS.)

(A) (15 PTS.) Show the following problem is NP-hard.

### Integer Linear Programming

**Instance:** A linear program in standard form, in which  $A$  and  $B$  contain only integers.

**Question:** Is there a solution for the linear program, in which the  $x$  must take integer values?

(B) (5 PTS.) A steel company must decide how to allocate next week's time on a rolling mill, which is a machine that takes unfinished slabs of steel as input and produce either of two semi-finished products: bands and coils. The mill's two products come off the rolling line at different rates:

Bands 200 tons/hr

Coils 140 tons/hr.

They also produce different profits:

Bands \$ 25/ton

Coils \$ 30/ton.

Based on current booked orders, the following upper bounds are placed on the amount of each product to produce:

Bands 6000 tons

Coils 4000 tons.

Given that there are 40 hours of production time available this week, the problem is to decide how many tons of bands and how many tons of coils should be produced to yield the greatest profit. Formulate this problem as a linear programming problem. Can you solve this problem by inspection?

- (C) (10 PTS.) A small airline, Ivy Air, flies between three cities: Ithaca (a small town in upstate New York), Newark (an eyesore in beautiful New Jersey), and Boston (a yuppie town in Massachusetts). They offer several flights but, for this problem, let us focus on the Friday afternoon flight that departs from Ithaca, stops in Newark, and continues to Boston. There are three types of passengers:
- Those traveling from Ithaca to Newark (god only knows why).
  - Those traveling from Newark to Boston (a very good idea).
  - Those traveling from Ithaca to Boston (it depends on who you know).

The aircraft is a small commuter plane that seats 30 passengers. The airline offers three fare classes:

- Y class: full coach.
- B class: nonrefundable.
- M class: nonrefundable, 3-week advanced purchase.

Ticket prices, which are largely determined by external influences (i.e., competitors), have been set and advertised as follows:

	Ithaca-Newark	Newark-Boston	Ithaca-Boston
Y	300	160	360
B	220	130	280
M	100	80	140

Based on past experience, demand forecasters at Ivy Air have determined the following upper bounds on the number of potential customers in each of the 9 possible origin-destination/fare-class combinations:

	Ithaca-Newark	Newark-Boston	Ithaca-Boston
Y	4	8	3
B	8	13	10
M	22	20	18

The goal is to decide how many tickets from each of the 9 origin/destination/fare-class combinations to sell. The constraints are that the plane cannot be overbooked on either the two legs of the flight and that the number of tickets made available cannot exceed the forecasted maximum demand. The objective is to maximize the revenue. Formulate this problem as a linear programming problem.

### 3. Strong duality. (30 PTS.)

Consider a directed graph  $G$  with source vertex  $s$  and target vertex  $t$  and associated costs  $c_e \geq 0$  on the edges. Let  $\mathcal{P}$  denote the set of all the directed (simple) paths from  $s$  to  $t$  in  $G$ .

Consider the following (very large) integer program:

$$\begin{aligned}
 & \text{minimize} && \sum_{e \in E(G)} c_e x_e \\
 & \text{subject to} && x_e \in \{0, 1\} \quad \forall e \in E(G) \\
 & && \sum_{e \in \pi} x_e \geq 1 \quad \forall \pi \in \mathcal{P}.
 \end{aligned}$$

- (5 PTS.) What does this IP compute?
- (5 PTS.) Write down the relaxation of this IP into a linear program.
- (10 PTS.) Write down the dual of the LP from (B). What is the interpretation of this new LP? What is it computing for the graph  $G$  (prove your answer)?
- (10 PTS.) The strong duality theorem states the following.

**Theorem 0.1** *If the primal LP problem has an optimal solution  $x^* = (x_1^*, \dots, x_n^*)$  then the dual also has an optimal solution,  $y^* = (y_1^*, \dots, y_m^*)$ , such that*

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*.$$

In the context of (A)-(C) what result is implied by this theorem if we apply it to the primal LP and its dual above? (For this, you can assume that the optimal solution to the LP of (B) is integral – which is not quite true – things are slightly more complicated than that.)