Chapter 33

Expanders III - The Zig Zag Product

By Sariel Har-Peled, April 23, 2018[1]

> Gradually, but not as gradually as it seemed to some parts of his brain, he began to infuse his tones with a sarcastic wounding bitterness. Nobody outside a madhouse, he tried to imply, could take seriously a single phrase of this conjectural, nugatory, deluded, tedious rubbish. Within quite a short time he was contriving to sound like an unusually fanatical Nazi trooper in charge of a book-burning reading out to the crowd excerpts from a pamphlet written by a pacifist, Jewish, literate Communist. A growing mutter, half-amused, half-indignant, arose about him, but he closed his ears to it and read on. Almost unconsciously he began to adopt an unnameable foreign accent and to read faster and faster, his head spinning. As if in a dream he heard Welch stirring, then whispering, then talking at his side. he began punctuating his discourse with smothered snorts of derision. He read on, spitting out the syllables like curses, leaving mispronunciations, omissions, spoonerisms uncorrected, turning over the pages of his script like a score-reader following a presto movement, raising his voice higher and higher. At last he found his final paragraph confronting him, stopped, and look at his audience.
>
> Kingsley Amis, Lucky Jim

33.1. Building a large expander with constant degree

33.1.1. Notations

For a vertex $v \in V(\mathsf{G})$, we will denote by $v_{\mathsf{G}}[i] = v[i]$ the ith neighbor of v in the graph G (we order the neighbors of a vertex in an arbitrary order).

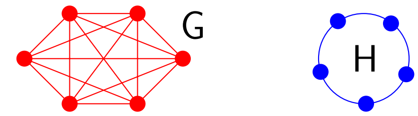
The regular graphs we next discuss have ***consistent labeling***. That is, for a regular graph G (we assume here that G is regular). This means that if u is the ith neighbor v then v is the ith neighbor of u. Formally, this means that $v[i][i] = v$, for all v and i. This is a non-trivial property, but its easy to verify that the low quality expander of Theorem 33.4.3 has this property. It is also easy to verify that the complete graph can be easily be made into having consistent labeling (exercise). These two graphs would be sufficient for our construction.

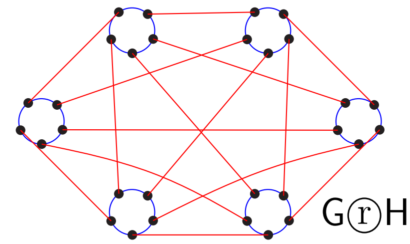
33.1.2. The Zig-Zag product

At this point, we know how to construct a good "small" expander. The question is how to build a large expander (i.e., large number of vertices) and with constant degree.

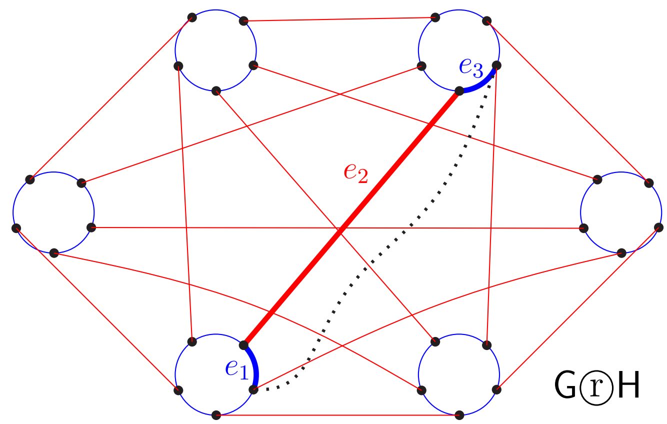
The intuition of the construction is the following: It is easy to improve the expansion qualities of a graph by squaring it. The problem is that the resulting graph G has a degree which is too large. To overcome this, we will replace every vertex in G by a copy of a small graph that is connected and has low degree. For example, we could replace every vertex of degree d in G by a path having d vertices. Every such vertex is now in charge of original edge of the graph. Naturally, such a replacement operation reduces the quality of the expansion of the resulting graph. In this case, replacing a vertex with a path is a potential "disaster", since every such subpath increases the lengths of the paths of the original graph by a factor of d (and intuitively, a good expander have "short" paths between any pair of vertices).

Consider a "large" (n, D)-graph G and a "small" (D, d)-graph H. As a first stage, we replace every vertex of G by a copy of H. The new graph K has $[\![n]\!] \times [\![D]\!]$ as a vertex set. Here, the edge $vu \in V(\mathsf{G})$, where $u = v[i]$ and $v = u[j]$, is replaced by the edge connecting $(v, i) \in V(\mathsf{K})$ with $(u, j) \in V(\mathsf{K})$. We will refer to this resulting edge $(v, i)(u, j)$ as a ***long*** edge. Also, we copy all the edges of the small graph to each one of its copies. That is, for each $i \in [\![n]\!]$, and $uv \in E(\mathsf{H})$, we add the edge $(i, u)(i, v)$ to K, which is a ***short*** edge. We will refer to K, which is a $(nD, d + 1)$-graph, as a ***replacement product*** of G and H, denoted by G ⓡ H. See figure on the right for an example.





Again, intuitively, we are losing because the expansion of the resulting graph had deteriorated too much. To overcome this problem, we will perform local shortcuts to shorten the paths in the resulting graph (and thus improve its expansion properties). A ***zig-zag-zig path*** in the replacement product graph K, is a three edge path $e_1 e_2 e_3$, where e_1 and e_3 are short edges, and the middle edge e_2 is a long edge. That is, if $e_1 = (i, u)(i, v)$, $e_2 = (i, v)(j, v')$, and $e_3 = (j, v')(j, u')$, then $e_1, e_2, e_3 \in E(\mathsf{K})$, $ij \in E(\mathsf{G})$, $uv \in E(\mathsf{H})$ and $v'u' \in E(\mathsf{H})$. Intuitively, you can think about e_1 as a small "zig" step in H, e_2 is a long "zag" step in G, and finally e_3 is a "zig" step in H.



Another way of representing a zig-zag-zig path $v_1 v_2 v_3 v_4$ starting at the vertex $v_1 = (i, v) \in V(\mathsf{F})$, is to parameterize it by two integers $\ell, \ell' \in [\![d]\!]$, where

$$v_1 = (i, v), \quad v_2 = (i, v_{\mathsf{H}}[\ell]) \quad v_3 = (i_{\mathsf{G}}[v_{\mathsf{H}}[\ell]], v_{\mathsf{H}}[\ell]) \quad v_4 = (i_{\mathsf{G}}[v_{\mathsf{H}}[\ell]]\ ,\ (v_{\mathsf{H}}[\ell])_{\mathsf{H}}[\ell']).$$

Let Z be the set of all (unordered) pairs of vertices of K connected by such a zig-zag-zig path. Note, that every vertex (i, v) of K has d^2 paths having (i, v) as an end point. Consider the graph $\mathsf{F} = (V(\mathsf{K}), Z)$. The graph F has nD vertices, and it is d^2 regular. Furthermore, since we shortcut all these zig-zag-zig paths in K, the graph F is a much better expander (intuitively) than K. We will refer to the graph F as the ***zig-zag*** product of G and H.

Definition 33.1.1. The ***zig-zag product*** of (n, D)-graph G and a (D, d)-graph H, is the (nD, d^2) graph $\mathsf{F} = \mathsf{G}$ ⓩ H, where the set of vertices is $[\![n]\!] \times [\![D]\!]$ and for any $v \in [\![n]\!]$, $i \in [\![D]\!]$, and $\ell, \ell' \in [\![d]\!]$ we have in F the edge connecting the vertex (i, v) with the vertex $(i_{\mathsf{G}}[v_{\mathsf{H}}[\ell]]\ ,\ (v_{\mathsf{H}}[\ell])_{\mathsf{H}}[\ell'])$.

Remark 33.1.2. We need the resulting zig-zag graph to have consistent labeling. For the sake of simplicity of exposition, we are just going to assume this property.

We next bound the tension of the zig-zag product graph.

Theorem 33.1.3. *We have* $\gamma(\mathsf{G}$ ⓩ $\mathsf{H}) \le \gamma_2(\mathsf{G})(\gamma_2(\mathsf{H}))^2$. *and* $\gamma_2(\mathsf{G}$ ⓩ $\mathsf{H}) \le \gamma_2(\mathsf{G})(\gamma_2(\mathsf{H}))^2$.

Proof: Let $\mathsf{G} = (\llbracket n \rrbracket, E)$ be a (n, D)-graph and $\mathsf{H} = (\llbracket D \rrbracket, E')$ be a (D, d)-graph. Fix any function $f : \llbracket n \rrbracket \times \llbracket D \rrbracket \to \mathbb{R}$, and observe that

$$\psi = \mathop{\mathbb{E}}_{\substack{u,v \in \llbracket n \rrbracket \\ k,\ell \in \llbracket D \rrbracket}} \left[|f(u,k) - f(v,\ell)|^2 \right] = \mathop{\mathbb{E}}_{k,\ell \in \llbracket D \rrbracket} \left[\mathop{\mathbb{E}}_{u,v \in \llbracket n \rrbracket} \left[|f(u,k) - f(v,\ell)|^2 \right] \right]$$

$$\leq \mathop{\mathbb{E}}_{k,\ell \in \llbracket D \rrbracket} \left[\gamma_2(\mathsf{G}) \mathop{\mathbb{E}}_{uv \in E(\mathsf{G})} \left[|f(u,k) - f(v,\ell)|^2 \right] \right] = \gamma_2(\mathsf{G}) \underbrace{\mathop{\mathbb{E}}_{k,\ell \in \llbracket D \rrbracket} \left[\mathop{\mathbb{E}}_{\substack{u \in \llbracket n \rrbracket \\ p \in \llbracket D \rrbracket}} \left[|f(u,k) - f(u[p],\ell)|^2 \right] \right]}_{=\Delta_1}.$$

Now,

$$\Delta_1 = \mathop{\mathbb{E}}_{\substack{u \in \llbracket n \rrbracket \\ \ell \in \llbracket D \rrbracket}} \left[\mathop{\mathbb{E}}_{k,p \in \llbracket D \rrbracket} \left[|f(u,k) - f(u[p],\ell)|^2 \right] \right] \leq \mathop{\mathbb{E}}_{\substack{u \in \llbracket n \rrbracket \\ \ell \in \llbracket D \rrbracket}} \left[\gamma_2(\mathsf{H}) \mathop{\mathbb{E}}_{kp \in E(\mathsf{H})} \left[|f(u,k) - f(u[p],\ell)|^2 \right] \right]$$

$$= \gamma_2(\mathsf{H}) \underbrace{\mathop{\mathbb{E}}_{\substack{u \in \llbracket n \rrbracket \\ \ell \in \llbracket D \rrbracket}} \left[\mathop{\mathbb{E}}_{\substack{p \in \llbracket D \rrbracket \\ j \in \llbracket d \rrbracket}} \left[|f(u,p[j]) - f(u[p],\ell)|^2 \right] \right]}_{=\Delta_2}.$$

Now,

$$\Delta_2 = \mathop{\mathbb{E}}_{\substack{j \in \llbracket d \rrbracket \\ \ell \in \llbracket D \rrbracket}} \left[\mathop{\mathbb{E}}_{\substack{u \in \llbracket n \rrbracket \\ p \in \llbracket D \rrbracket}} \left[|f(u,p[j]) - f(u[p],\ell)|^2 \right] \right] = \mathop{\mathbb{E}}_{\substack{j \in \llbracket d \rrbracket \\ \ell \in \llbracket D \rrbracket}} \left[\mathop{\mathbb{E}}_{\substack{v \in \llbracket n \rrbracket \\ p \in \llbracket D \rrbracket}} \left[|f(v[p],p[j]) - f(v,\ell)|^2 \right] \right]$$

$$= \mathop{\mathbb{E}}_{\substack{j \in \llbracket d \rrbracket \\ v \in \llbracket n \rrbracket}} \left[\mathop{\mathbb{E}}_{\substack{p \in \llbracket D \rrbracket \\ \ell \in \llbracket D \rrbracket}} \left[|f(v[p],p[j]) - f(v,\ell)|^2 \right] \right]$$

$$= \gamma_2(\mathsf{H}) \underbrace{\mathop{\mathbb{E}}_{\substack{j \in \llbracket d \rrbracket \\ v \in \llbracket n \rrbracket}} \left[\mathop{\mathbb{E}}_{p\ell \in E(\mathsf{H})} \left[|f(v[p],p[j]) - f(v,\ell)|^2 \right] \right]}_{=\Delta_3}.$$

Now, we have

$$\Delta_3 = \mathop{\mathbb{E}}_{\substack{j \in \llbracket d \rrbracket \\ v \in \llbracket n \rrbracket}} \left[\mathop{\mathbb{E}}_{\substack{p \in \llbracket D \rrbracket \\ i \in \llbracket d \rrbracket}} \left[|f(v[p],p[j]) - f(v,p[i])|^2 \right] \right] = \mathop{\mathbb{E}}_{(u,k)(\ell,v) \in E(\mathsf{G} \,ⓩ\, \mathsf{H})} [|f(u,k) - f(\ell,v)|],$$

as $(v[p], p[j])$ is adjacent to $(v[p], p)$ (a short edge), which is in turn adjacent to (v, p) (a long edge), which is adjacent to $(v, p[i])$ (a short edge). Namely, $(v[p], p[j])$ and $(v, p[i])$ form the endpoints of a zig-zag path in the replacement product of G and H. That is, these two endpoints are connected by an edge in the zig-zag product graph. Furthermore, it is easy to verify that each zig-zag edge get accounted for in this representation exactly once, implying the above inequality. Thus, we have $\psi \leq \gamma_2(\mathsf{G})(\gamma_2(\mathsf{H}))^2 \Delta_3$, which implies the claim.

The second claim follows by similar argumentation. ■

33.1.3. Squaring

The last component in our construction, is ***squaring***squaring!graph a graph. Given a (n, d)-graph G, consider the multigraph G^2 formed by connecting any vertices connected in G by a path of length 2. Clearly, if M is the adjacency matrix of G, then the adjacency matrix of G^2 is the matrix M^2. Note, that $\left(\mathsf{M}^2\right)_{ij}$ is the number of distinct paths of length 2 in G from i to j. Note, that the new graph might have self loops, which does not effect our analysis, so we keep them in.

Lemma 33.1.4. *Let G be a (n, d)-graph. The graph G^2 is a (n, d^2)-graph. Furthermore $\gamma_2\left(\mathsf{G}^2\right) = \frac{(\gamma_2(\mathsf{G}))^2}{2\gamma_2(\mathsf{G})-1}$.*

Proof: The graph G^2 has eigenvalues $\left(\widehat{\lambda}_1(\mathsf{G})\right)^2, \ldots, \left(\widehat{\lambda}_1(\mathsf{G})\right)^2$ for its matrix Q^2. As such, we have that

$$\widehat{\lambda}\left(\mathsf{G}^2\right) = \max\left(\widehat{\lambda}_2\left(\mathsf{G}^2\right), -\widehat{\lambda}_n\left(\mathsf{G}^2\right)\right).$$

Now, $\widehat{\lambda}_1\left(\mathsf{G}^2\right) = 1$. Now, if $\widehat{\lambda}_2(\mathsf{G}) \geq \left|\widehat{\lambda}_n(\mathsf{G})\right| < 1$ then $\widehat{\lambda}\left(\mathsf{G}^2\right) = \widehat{\lambda}_2\left(\mathsf{G}^2\right) = \left(\widehat{\lambda}_2(\mathsf{G})\right)^2 = \left(\widehat{\lambda}(\mathsf{G})\right)^2$.

If $\widehat{\lambda}_2(\mathsf{G}) < \left|\widehat{\lambda}_n(\mathsf{G})\right|$ then $\widehat{\lambda}\left(\mathsf{G}^2\right) = \widehat{\lambda}_2\left(\mathsf{G}^2\right) = \left(\widehat{\lambda}_n(\mathsf{G})\right)^2 = \left(\widehat{\lambda}(\mathsf{G})\right)^2$..

Thus, in either case $\widehat{\lambda}\left(\mathsf{G}^2\right) = \left(\widehat{\lambda}(\mathsf{G})\right)^2$. Now, By Lemma 33.4.1 $\gamma_2(\mathsf{G}) = \frac{1}{1-\widehat{\lambda}(\mathsf{G})}$, which implies that $\widehat{\lambda}(\mathsf{G}) = 1 - 1/\gamma_2(\mathsf{G})$, and thus

$$\gamma_2\left(\mathsf{G}^2\right) = \frac{1}{1 - \widehat{\lambda}(\mathsf{G}^2)} = \frac{1}{1 - \left(\widehat{\lambda}(\mathsf{G})\right)^2} = \frac{1}{1 - \left(1 - \frac{1}{\gamma_2(\mathsf{G})}\right)^2} = \frac{\gamma_2(\mathsf{G})}{2 - \frac{1}{\gamma_2(\mathsf{G})}} = \frac{(\gamma_2(\mathsf{G}))^2}{2\gamma_2(\mathsf{G}) - 1}.$$

■

33.1.4. The construction

So, let build an expander using Theorem 33.4.3, with parameters $r = 7$ $q = 2^4 = 32$. Let $d = q^2 = 256$. The resulting graph H has $N = q^{r+1} = d^4$ vertices, and it is $d = q^2$ regular. Furthermore, $\widehat{\lambda}_i \leq r/q = 7/32$, for all $i \geq 2$. As such, we have

$$\gamma(\mathsf{H}) = \gamma_2(\mathsf{H}) = \frac{1}{1 - 7/32} = \frac{32}{25}.$$

Let G_0 be any graph that its square is the complete graph over $n_0 = N + 1$ vertices. Observe that G_0^2 is d^4-regular. Set $\mathsf{G}_i = \left(\mathsf{G}_{i-1}^2 ⓩ \mathsf{H}\right)$, Clearly, the graph G_i has

$$n_i = n_{i-1}N$$

vertices. The graph $\mathsf{G}_{i-1}^2 ⓩ \mathsf{H}$ is d^2 regular. As far as the bi-tension, let $\alpha_i = \gamma_2(\mathsf{G}_i)$. We have that

$$\alpha_i = \frac{\alpha_{i-1}^2}{2\alpha_{i-1} - 1}(\gamma_2(\mathsf{H}))^2 = \frac{\alpha_{i-1}^2}{2\alpha_{i-1} - 1}\left(\frac{32}{25}\right)^2 \leq 1.64\frac{\alpha_{i-1}^2}{2\alpha_{i-1} - 1}.$$

It is now easy to verify, that α_i can not be bigger than 5.

Theorem 33.1.5. *For any $i \geq 0$, one can compute deterministically a graph G_i with $n_i = (d^4 + 1)d^{4i}$ vertices, which is d^2 regular, where $d = 256$. The graph G_i is a $(1/10)$-expander.*

Proof: The construction is described above. As for the expansion, since the bi-tension bounds the tension of a graph, we have that $\gamma(\mathsf{G}_i) \leq \gamma_2(\mathsf{G}_i) \leq 5$. Now, by Lemma 33.4.2, we have that G_i is a δ-expander, where $\delta \geq 1/(2\gamma(\mathsf{G}_i)) \geq 1/10$. ■

33.2. Bibliographical notes

A good survey on expanders is the monograph by Hoory *et al.* [HLW06]. The small expander construction is from the paper by Alon *et al.* [ASS08] (but its originally from the work by Along and Roichman [AR94]). The work by Alon *et al.* [ASS08] contains a construction of an expander that is constant degree, which is of similar complexity to the one we presented here. Instead, we used the zig-zag expander construction from the influential work of Reingold *et al.* [RVW02]. Our analysis however, is from an upcoming paper by Mendel and Naor [MN08]. This analysis is arguably reasonably simple (as simplicity is in the eye of the beholder, we will avoid claim that its the simplest), and (even better) provide a good intuition and a systematic approach to analyzing the expansion.

We took a creative freedom in naming notations, and the name tension and bi-tension are the author's own invention.

33.3. Exercises

Exercise 33.3.1 (Expanders made easy.). By considering a random bipartite three-regular graph on $2n$ vertices obtained by picking three random permutations between the two sides of the bipartite graph, prove that there is a $c > 0$ such that for every n there exits a $(2n, 3, c)$-expander. (What is the value of c in your construction?)

Exercise 33.3.2 (Is your consistency in vain?). In the construction, we assumed that the graphs we are dealing with when building expanders have consistent labeling. This can be enforced by working with bipartite graphs, which implies modifying the construction slightly.

(A) Prove that a d-regular bipartite graph always has a consistent labeling (hint: consider matchings in this graph).
(B) Prove that if G is bipartite so is the graph G^3 (the cubed graph).
(C) Let G be a (n, D)-graph and let H be a (D, d)-graph. Prove that if G is bipartite then $\mathsf{G}\mathsf{G} \text{ⓩ} \mathsf{H}$ is bipartite.
(D) Describe in detail a construction of an expander that is: (i) bipartite, and (ii) has consistent labeling at every stage of the construction (prove this property if necessary). For the ith graph in your series, what is its vertex degree, how many vertices it has, and what is the quality of expansion it provides?

Exercise 33.3.3 (Tension and bi-tension.). [**30 points**]

Disprove (i.e., give a counter example) that there exists a universal constant c, such that for any connected graph G, we have that $\gamma(\mathsf{G}) \le \gamma_2(\mathsf{G}) \le c\gamma(\mathsf{G})$.

Acknowledgments

Much of the presentation was followed suggestions by Manor Mendel. He also contributed some of the figures.

33.4. From previous lectures

Lemma 33.4.1. *Let $\mathsf{G} = (V, E)$ be a connected d-regular graph with n vertices. Then $\gamma_2(\mathsf{G}) = \dfrac{1}{1 - \widehat{\lambda}}$, where $\widehat{\lambda} = \widehat{\lambda}(\mathsf{G})$, where $\widehat{\lambda}(\mathsf{G}) = \max\left(\widehat{\lambda}_2, -\widehat{\lambda}_n\right)$, where $\widehat{\lambda}_i$ is the ith largest eigenvalue of the random walk matrix associated with G.*

Lemma 33.4.2. *Let $\mathsf{G} = (V, E)$ be a given connected d-regular graph with n vertices. Then, G is a δ-expander, where $\delta \geq \dfrac{1}{2\gamma(\mathsf{G})}$ and $\gamma(\mathsf{G})$ is the tension of G.*

Theorem 33.4.3. *For any $t > 0, r > 0$ and $q = 2^t$, where $r < q$, we have that $\mathsf{LD}(q, r)$ is a graph with q^{r+1} vertices. Furthermore, $\lambda_1(\mathsf{LD}(q, r)) = q^2$, and $\lambda_i(\mathsf{LD}(q, r)) \leq rq$, for $i = 2, \ldots, n$.*

In particular, if $r \leq q/2$, then $\mathsf{LD}(q, r)$ is a $\left[q^{r+1}, q^2, \frac{1}{4}\right]$-expander.

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