CS 497 — Randomized Algorithms

Fall, 2002

Homework 5, Due 12/3/02

1 Homework

1. *From Magnifiers to Expanders.*

   [5 Points]

   Definition 1.1 An (n, k, c)-*expander* is an X, Y-bipartite G with $|X| = |Y| = n$ such that $\Delta(G) \leq k$ and that $|N(S)| \geq (1 + c(1 - |S|/n)) \cdot |S|$ for every $S \subseteq X$ with $|S| \leq n/2$. An (n, k, c)-*magnifier* is an n-vertex graph G such that $\Delta(G) \leq k$ such that $\left|N(S) \cap \overline{S}\right| \geq c \cdot |S|$ for every $S \subseteq V(G)$ with $|S| \leq n/2$.

   Note that a magnifier is not necessarily bipartite.

   Let G be an (n, k, c)-magnifier with vertices $v_1, \ldots, v_n$. Let H be the bipartite graph with partite sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ such that $x_i y_j \in E(H)$ if and only if $i = j$ or $v_i v_j \in E(G)$. Prove that H is an $(n, k+1, c)$-expander.

2. *Existence of expanders of linear size.*

   [20 Points]

   (a) Let X be a random variable giving the size of the union of k s-subsets of $[n] = \{1, \ldots, n\}$ chosen at random from $\binom{[n]}{s} = \left\{ S \;\middle|\; S \subseteq [n], |S| = s \right\}$. Prove that

   $$\mathbf{Pr}[X \leq l] \leq \binom{n}{l} \left(\frac{l}{n}\right)^{ks}.$$

   (b) For $\alpha\beta < 1$, prove that there is a constant k such that, when n is sufficiently large, there exists a subgraph of $K_{n,n}$ with maximum degree at most k such that $|N(S)| \geq \beta |S|$ whenever $|S| \leq \alpha n$. (Hint: Generate bipartite subgraphs of $K_{n,n}$ by taking the union of k random perfect matchings.)

   (c) Conclude the exists of K such that (n, k, c)-expanders exists for all sufficiently large n.

3. *Automorphism and Eigenvalues.*

   [20 Points]

Definition 1.2 An *isomorphism* from G to H is bijection f that maps $V(G)$ to $V(H)$ and $E(G)$ to $E(H)$, such that each edge of G with endpoints u and v is mapped to an edge with en points $f(u)$ and $f(v)$.

An *automorphism* of G is an isomorphism from G to G.

(a) Prove that σ is an automorphism of G if and only if the permutation matrix corresponding to σ commutes with the adjacency matrix of G; that is $PA = AP$.

(b) Let X be an eigenvector of G for an eigenvalue of multiplicity 1, and let P be the permutation matrix for an automorphism of G. Prove that $Px = \pm x$.

(c) Conclude that when every eigenvalue of G has multiplicity 1, every automorphism of G is an involution, meaning that repeating it yields the identity

4. *The eigenvalues of a cycle.*

[20 Points]

Show that the eigenvalues of C_n (the cycle of length n) are $2, 2\cos(2\pi/n), \cos(4\pi/n), \ldots, 2\cos(2(n-1)\pi/n)$. Thus if n is odd, 2 has multiplicity one and each other eigenvalue has multiplicity two; if n is even, each of 2 and -2 has multiplicity one, and every other eigenvalue has multiplicity two. [Hint: Assuming as we may, that $V(C_n) = \mathbb{Z}_n$ and $E(C_n) = \left\{ ij \;\middle|\; i - j = \pm 1 \right\}$, note that $(1, \omega, \omega^2, \ldots, \omega^{n-1})$ is the eigenvector of each n-th root of unity ω.]

5. *On the Importance of Being Empty.*

[20 Points]

(Due to Jeff Erickson)

Let P be a set of n points picked randomly, uniformly and independently from $\mathcal{I}$, where $\mathcal{I} = [0,1] \times [0,1]$ is the unit square in the plane. A point $p = (x, y) \in P$ is a *corner*, if the triangle $\triangle pqr$ does not contain any point of P in its interior, where $q = (x, 0)$ and $r = (0, 0)$.

Let X be the number of corners in P. Prove a bound, as tight as possible (asymptoticly, constants are not important), on the value of $\mathbf{E}[X]$. The shorter your solution is, and the more elegant it is, the more points you would get for your solution. More points would be given to solutions avoiding the usage of integrals.

6. *VC Dimension.*

[10 Points]

(a) What is the VC dimension of (X, R), where X is all the points in the plane, and R is the set of all polygons in the plane?

(b) Bound the VC dimension of (X, R) in the plane, where X is the set of all points in the plane, and R is the set of all ellipses. (An ellipse is the image of a circle under affine transformation.) Prove your bound. The bound should be as tight as possible.

(c) Bound the VC dimension of simple polygons in the plane having k sides. Again, prove your bound.

7. *Going to infinity, and never coming back.*

   [10 Points]

   The rooted binary tree is an infinite graph T with one distinguished vertex R from which comes a single edge; at every other vertex there are three edges and there are no closed loops. The random walk on T jumps from a vertex along each available edge with equal probability. Show the the random walk is transient.

8. *Take those Flowers.*

   [10 Points]

   An opera singer is due to perform a long series of concerts. Having a fine artistic temperament, he is liable to pull out each night with probability 1/2. Once this has happened he will not sing again until the promoter convince him of her high regard. This she does by sending flowers every day until the singer returns. Flower costing x thousand pounds, $0 \leq x \leq 1$, bring about a reconciliation with probability $\sqrt{x}$. The promoter stands to make 750 pounds from each successful concert. How much should she spend on flowers?

Guidelines

Homeworks are done in singletons, and I expect you to work on them **independently**. If you copy a solution from somewhere (web, class notes, books, the universe at large, your brain), etc, indicate it in LARGE FRIENDLY LETTERS on your exercise.

If you have any questions, feel free to come to my office hours (Thursday 16:00-17:00). Please email me if you plan to come, so I would be sure to be around. Also, I would like to encourage you to send questions to the newsgroup (`uiuc.class.cs497shp`).

It would be nice of you to type your homeworks, but this is not required, and would not result in any extra credit.